CAMBRIDGE SCHOOL AND COLLEGE TEXT BOOKS.

ELEMENTS OF ALGEBRA.

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ALGEBRA.

CHAPTER I.

DEFINITIONS, SIGNS, &c.

1.—In Arithmetic, numbers are represented by means of *figures*, each figure having always one particular value; but in Algebra, they are also represented by means of other symbols, generally letters, and each letter, while it has the same meaning throughout any one operation, may be considered to represent any number whatever.

Thus, the figure 5 always denotes the particular number *five;* but the letter a may be understood to have any assigned value, or it may be regarded as a short expression for "any number."

2.—Operations to be performed on numbers, or with them, are indicated by *symbols* (i. e. *signs*), or by symbolical arrangements.

The simple arithmetical operations are indicated as follows:—

Addition by the sign $+$, which is named *plus;* thus $a + b$ indicates the result when b has been added to a.

Subtraction by $-$, which is named *minus*, and indicates that the first number is to have the second subtracted from it; thus $a - b$ is the result when b has been subtracted from a.

The excess of the greater over the less of two numbers, when it is not known which is the greater, is indicated by $\sim$; thus $x \sim y$ means $x - y$ or $y - x$, according as x or y may be found to be the greater.

Multiplication is represented either by $\times$, or by $.$, or by

writing the letters consecutively; thus $a \times b$, $a \,.\, b$, ab, all represent the product of a and b.

In employing figures the first method is generally preferable, on account of · being used as a decimal point; and because confusion would arise, in using the third method, between this and the ordinary arithmetical meaning assigned to two or more figures written together.

Division is represented by $\div$, or by a fractional form, in which the dividend is placed in the numerator, and the divisor in the denominator; thus $a \div b$ and $\frac{a}{b}$ both represent the quotient obtained by dividing a by b.

$=$ is used to indicate the equality of two numbers between which it is placed.

$>$ means *is greater than;* $<$ *is less than.*

$\because$ means *because,* $\therefore$ *therefore.*

3.—The signs $+$ and $-$, besides being placed before the second of two numbers, to indicate that it is to be added to the first or subtracted from it, are also placed before a single number to indicate whether it is *additive* or *subtractive;* i. e. whether, *if it were* combined with another, it *would be* added to it or subtracted from it.

Instead of *additive* and *subtractive,* it is usual to designate a number as *positive* or *negative.*

> Thus if x be equal to $+3$, x taken with another number, as 5, will make 8; but if $x = -3$, it will, when taken with 5, make 2.

If a number have no sign placed before it, it is to be considered positive.

A negative number has therefore, when combined in this manner with another number, an effect opposite to that of a positive number, and may be regarded as the reverse of a positive number. This meaning of a negative may be illustrated by its application to concrete quantities. For instance, if a line measured from a given point in one direction be considered to have a positive length, a negative length of the same line will be a length measured from that point in the opposite direction. Negative time from a given instant will mean time measured backwards from

that instant, i. e. before it. A man who is £100 in debt might be said to be worth −£100, &c.

4.—When several of the same letters are taken together, their sum is indicated by writing the number of them before the letter. Thus $6a$ means six a's added together, or 6 times a. This number is called a *coefficient*.

The positive or negative sign of a quantity will be written before the coefficient, as $-4x$.

A letter with no coefficient expressed will be understood to be taken *once*, i. e. to have 1 for its coefficient; thus a and $1a$ have the same meaning.

A figure may be written before a product as its coefficient; as $3ab$, which means not $3a \times 3b$, but 3 times ab.

> Coefficients were introduced about A.D. 1600; before that time the letter was repeated several times; as $x+x+x$, instead of $3x$.

A *letter* may be used before another letter or before a product as its coefficient; thus ax may mean x taken a times, a being therefore the coefficient of x.

5.—When several of the same letters are multiplied together, their product is indicated by writing the number of them above the letter to the right. Thus a^3 means that three a's are to be multiplied together. This figure is called the *Index* or *Exponent* of the letter.

A letter with no index expressed will be understood to have the index 1, for a and a^1 have the same meaning.

Before indices were introduced, the letter was repeated; as $a \times a \times a \times a$, or $aaaa$, instead of a^4.

The product of several of the same numbers is called a *power* of that number; thus a^2, a^5, are called the second and fifth powers of a. The second and third powers of a number are also called its *square* and *cube*.

An index (unless a bracket is used) belongs only to the letter over which it is placed; thus ab^2c indicates the product of a, b^2, and c, the b only being squared.

A *letter* may be used as an index; thus x^r means r x's multiplied together.

6.—An algebraical quantity made up of several letters is called an *expression.*

A letter or product in an expression connected with the rest by the sign + or − is called a *term*. Thus in the expression $2a-3b^2+4a^2b$, $2a$, $3b^2$, $4a^2b$ are the terms.

An expression consisting of one term is called a *simple* expression.

If it consist of more terms than one, it is called a *compound* expression.

A compound expression consisting of two terms is said to be *binomial*; of three, *trinomial*; of several, *multinomial.*

Similarly, a simple expression is sometimes called a *mononomial* expression.

7.—Since a term is the product of the letters composing it, those letters may be called its *factors.*

The sum of the indices of the letters in a term is often called the *number of dimensions* of the term. Thus $4ab^2c^3$, in which 1, 2, 3 are the indices, is of *six* dimensions.

It is sometimes called the *order* or *degree* of the term: the above would be said to be of the *sixth order*, or of the *sixth degree.*

If all the terms of an expression are of the same dimensions, the expression is said to be *homogeneous*; as $2a^4-3a^3b+ab^3-2abc^2$.

Terms are said to be *like* when they consist of the same letters raised severally to the same powers; as $2a^2xy^3$ and $-5a^2xy^3$. But $2a^2xy^3$ and $5a^2x^3y$ are *unlike*, since, although their letters are the same, the powers to which they are severally raised are different.

8.—The arrangement of the letters composing a term will not affect the numerical value of the term; thus $2a^3xy^2$ and $2xy^2a^3$ have the same value. But it is generally best to write them in the alphabetical order, $2a^3xy^2$ being preferable to $2xy^2a^3$ or $2xa^3y^2$, &c.

Similarly, the terms of an expression may be arranged in any order without affecting its value, provided that each term has its proper sign prefixed: thus $a-3b+2c-8d$ is the same as $a+2c-3b-8d$, or as $-3b+a-8d+2c$. But an alphabetical arrangement is generally to be preferred.

Terms in which the same letter recurs in different powers should be arranged according to those powers, either in an ascending or descending order.

Thus, instead of $ax^3-2a^3x-4a^2x^2+x^4+7a^4$,
it will be better to write $x^4+ax^3-4a^2x^2-2a^3x+7a^4$,
or else $7a^4-2a^3x-4a^2x^2+ax^3+x^4$;

the former according to the descending powers of x, the latter according to its ascending powers.

Exercise 1.

(1) Write down the coefficients of the terms of the expression $2a+3b+c+4d+e$; and the indices of $5a^2bc^3d$ and of $-xy^3z^5$.

(2) Write down the sum of $2x$ and $3y$; and their difference (the latter being taken from the former).

If $a=1$, $b=2$, $c=6$, $d=3$, $e=0$, find the values of

(3) $5a$, $3c$, $4e$, $2bc$, $7ad$, abc, $-bcd$, abe, $-3acd$.

(4) $a+b+c$, $3b+c+4d$, $5a-2b+c-2e$, $3a+4c-9d$.

(5) $2ab-ad+4bc-2cd$, $6ac-2bd-7ad+8be$.

(6) $2ab^2$, $10a^2bc$, $2ab^2c^3$, a^4b^3d, $2ac^3-bd^3$, $2a^2b^2c-a^3b^2d^2$.

(7) $abc^2-b^2cd+a^2cd-b^2de$, $3a^2cd-2ab^2c-9ac^2e+3a^2bd$.

(8) Which are the like terms in the expression—

$$2a^2xy-4a^2bx-2ax^2y+a^2bx+5axy^2-2bx^2y\,?$$

(9) Of how many dimensions are $2a^2b^3x^2y^4$, $3abx^3y^2$, $-4abxy$, $5a^3$, $-a^2xy^3$?

(10) Which terms are homogeneous in the expression—

$$5a^3xy-2bx^2y^2-a^3xy^2+a^2by-7a^3b^3x+bx^4\,?$$

If $a=3$, $b=2$, $x=4$, $y=1$, find the values of

(11) $\dfrac{2a-b}{x-2y}$, $\dfrac{a+2x}{3a-b+4y}$, $\dfrac{10x}{7y}$, $\dfrac{a+b-x-y}{2a-5y}$.

(12) a^x, a^xb^y, $10x^b$, $2a^x-10b^x$, $\dfrac{x^a}{8y^b}$, $\dfrac{x^b-y^b}{x^a-y^a}$.

(13) Write down the product of the square of a and the cube of b; and twice the product of a, the cube of x, and the square of y.

(14) Write down, in a fractional form, the quotient of $2a$

divided by $3b$; of $3x$ divided by $7y$; and of $2a+5x$ divided by $3b-2y$.

(15) Write down all the products of three dimensions which can be formed of x and y and their powers.

(16) Find the sum of all the terms of two dimensions, which can be formed of a, b, c, and their powers, when $a=3$, $b=2$, $c=1$.

(17) Determine the value of $a^x \sim b^y$ when $a=8$, $b=6$, $x=3$, $y=4$.

(18) Find the value of $3ax-2by+xy$, when $a=3$, $b=2$, $x=\frac{1}{2}$, $y=\frac{1}{3}$.

(19) Find the value of $-5x^3y+2x^2y^2+3xy^3+9y^4$, when $x=\frac{1}{2}$, $y=\frac{1}{3}$.

(20) What will be the quotient when the sum of the cubes of a, b, x, y is divided by the sum of the simple powers, if $a=2$, $b=\frac{1}{2}$, $x=5$, $y=\frac{1}{8}$?

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION.

ADDITION.

9.—(*a*) UNLIKE terms can only be added by being written down separately, and connected by the sign + or —. Thus the sum of $2a$ and $3b$ can be written only as $2a + 3b$, or $3b + 2a$. The sum of $4x$ and $-5y$ will be $4x - 5y$, or, $-5y + 4x$. The sum of $-2a^2b$ and $-4ab^2$ will be $-2a^2b - 4ab^2$, or $-4ab^2 - 2a^2b$.

It will be noticed that *addition* has in Algebra a sense rather more extended than in Arithmetic, and includes, if one of the quantities is negative, what is really subtraction.

(β) *Like* terms, if they have the same sign, will have for their sum a like term having also the same sign, and whose coefficient is the sum of their coefficients.

Thus the sum of $5a$ and $2a$ is $7a$;

the sum of $-3a^2b$, $-2a^2b$, and $-a^2b$ is $-6a^2b$.

(γ) Two *like* terms, having different signs, will have for their sum a like term, whose coefficient is the difference between their coefficients, and whose sign is that of the greater.

Thus the sum of $5xy^2$ and $-2xy^2$ is $3xy^2$;

the sum of $7ab^2c$ and $-9ab^2c$ is $-2ab^2c$.

(δ) The sum of several like terms, whose signs are different, will be obtained by finding (by β) the sums of the positive and negative terms, and adding these (by γ).

Thus the sum of $3ax$, $-5ax$, $-ax$, and $12ax$, is $9ax$;

the sum of $2ab^3$, $-5ab^3$, ab^3, and $-4ab^3$, is $-6ab^3$.

(ϵ) Two or more compound expressions will be added together by adding their separate terms. The simplest arrangement will be made by placing them one under the other, so that their like terms may fall in columns; each column will then be added in the manner stated above.

Ex. (1)

$$\begin{array}{rrrr} 2a^3 - & 3a^2b + & 4ab^2 + & b^3 \\ a^3 + & 4a^2b - & 7ab^2 - & 2b^3 \\ -3a^3 + & a^2b - & 3ab^2 - & 4b^3 \\ 2a^3 + & 2a^2b + & 6ab^2 - & 3b^3 \\ \hline 2a^3 + & 4a^2b & & -8b^3 \end{array}$$

Ex. (2)

$$\begin{array}{rrrrr} -x^3 - & 2x^2y & & +6y^3 & -1 \\ -3x^3 - & 4x^2y + & 2xy^2 & & +5 \\ -4x^3 + & 6x^2y & & & +2 \\ & x^2y & & -y^3 & \\ & -2x^2y & & & -5 \\ \hline -8x^3 - & x^2y + & 2xy^2 + & 5y^3 & +1 \end{array}$$

Exercise 2.

Find the sums of—

(1) $3a$, $4a$, $5a$, and $7a$; of $2x$, $5x$, $7x$, and $-6x$; of $-5y$, $12y$, y, and $3y$.

(2) $5a^2b$, $-2a^2b$, $5a^2b$, and $-7a^2b$; of $4xy^2$, $-12xy^2$, $2xy^2$, and xy^2; of $-a^3bc$, $4a^3bc$, $-7a^3bc$, and $-5a^3bc$.

(3) $2a+b+3c$, $3a+4b-5c$, $a-2b+6c$, $2a-b+c$.

(4) $x+y+z$, $2x-3y+2z$, $-4x+2y-5z$, $2x-2y+2z$.

(5) $2a^2b-3ab^2-b^3$, $5a^2b-ab^2+2b^3$, $-ab^2+b^3$, a^2b-2ab^2, $2a^2b-b^3$.

(6) $5a^4-3a^3x-2a^2x^2+7ax^3+x^4$, $-a^4-3a^2x^2-x^4$, $-4a^4+3a^3x-ax^3$.

(7) $c^4-3c^3+2c^2-4c+7$, $2c^4+3c^3+2c^2+5c+6$, $-4c^4-4c^2-5$.

(8) $3x^2y-4y^3$, $2x^3-5x^2y$, $7x^2y-7xy^2$, $-x^3-y^3$, $-5x^2y+7xy^2+4y^3$.

(9) $3x^2-xy+xz-3y^2-4yz-z^2$, $-5x^2-xy-xz+5yz$, $6x^2-6y^2-6z^2$, $4xz-5yz+3z^2$, $-4x^2+y^2+3yz+3z^2$.

(10) $2a^4-4a^2x^2-x^4$, $3a^3x-2ax^3-7x^4$,
$-a^4-5a^3x+6a^2x^2+ax^3$, $-8a^2x^2-5ax^3+6x^4$,
$-a^4+12a^3x+6a^2x^2$, $2a^3x-3ax^3+2x^4$.

(11) $m^5-3m^4n-6m^3n^2$, $-5m^4n+m^3n^2+m^2n^3$,

$7m^3n^2+4m^2n^3-3mn^4, -2m^2n^3-3mn^4+4n^5, 2mn^4+2n^5+3m^5,$
$-n^5+2m^5+7m^4n.$

(12) $a^2x^2-2abxy+ax+2c^2,\ -3b^2y^2-3abxy-4by-3c^2,$
$3a^2x^2-4b^2y^2-2ax+5by;\ -4a^2x^2+ax+c^2,\ 2b^2y^2-4abxy-by.$

SUBTRACTION.

10.—SUBTRACTION is the reverse of Addition, i.e. a negative Addition, and is to be performed by changing the signs of the subtractive terms, and adding.

Thus the result when $2a$ is subtracted from $5a$ is

$$5a-2a, \text{ i.e. } 3a.$$

When $-2a$ is subtracted from $5a$, the result is

$$5a+2a, \text{ i.e. } 7a.$$

When one compound expression is to be subtracted from another, its terms may be put under the like terms of the other, as in Addition. The signs of the lower line may then, before adding, be either actually changed, or (as is more convenient in application afterwards) they may be *considered as changed.*

Ex. (1) From $4x^3-3x^2y-xy^2+2y^3$
take $2x^3-x^2y+5xy^2-3y^3$.

Changing the signs of the latter expression,

$$\begin{array}{rrrr} 4x^3 & -\ 3x^2y & -\ xy^2 & +\ 2y^3 \\ -\ 2x^3 & +\ x^2y & -\ 5xy^2 & +\ 3y^3 \\ \hline 2x^3 & -\ 2x^2y & -\ 6xy^2 & +\ 5y^3 \end{array} \quad \textit{Ans.}$$

Ex. (2) From $3a^4-2a^3b-b^4$ take $a^4-2a^3b+4a^2b^2-ab^3$.

$$\begin{array}{rrrrr} 3a^4 & -\ 2a^3b & & & -\ b^4 \\ -\ a^4 & +\ 2a^3b & -\ 4a^2b^2 & +\ ab^3 & \\ \hline 2a^4 & & -\ 4a^2b^2 & +\ ab^3 & -\ b^4 \end{array} \quad \textit{Ans.}$$

Ex. (3) From $a^3x^2+2a^2x^3-4ax^4$
take $a^5+4a^3x^2-3a^2x^3-4ax^4$.

Considering the signs to be changed before adding,

$$\begin{array}{rrrr} & a^3x^2 & +\ 2a^2x^3 & -\ 4ax^4 \\ a^5 & +\ 4a^3x^2 & -\ 3a^2x^3 & -\ 4ax^4 \\ \hline -\ a^5 & -\ 3a^3x^2 & +\ 5a^2x^3 & \end{array} \quad \textit{Ans.}$$

Exercise 3.

(1) Take $3a$ from $5a$; $6a$ from $3a$; $-2a$ from $4a$; $-8a$ from a; $4a$ from $-2a$; and a from $-a$.

(2) From $2a-3b+c$ take $a-4b-c$; and from $4ab-2bc$ take $5ab+3bc-4ac$.

(3) From $x^2-5xy+xz-y^2+7yz+2z^2$
take $x^2-xy-xz+2yz+3z^2$.

(4) From $2ax^2+3abx-4b^2x+12b^3$
take $ax^2-4abx+bx^2-5b^2x-x^3$.

(5) Take $8x^3-7x^2y+xy^2-y^3+9x^2-xy+6y^2-4$
from $6x^3-7x^2y+4xy^2-2y^3-5x^2+xy-4y^2+2$.

(6) From a^4-b^4 take $4a^3b-6a^2b^2+4ab^3$; and from the result take $2a^4-4a^3b+6a^2b^2+4ab^3-2b^4$.

(7) From $x^3y^2-3x^2y^3+4xy^4-y^5$ take $-x^5+2x^4y-4xy^4-4y^5$; add the same two expressions together, and subtract the former result from the latter.

(8) Take $2a^2bc-5ab^2c+2abc^2-5b^2c^2$ from
$a^2b^2-a^2bc-8ab^2c-a^2c^2+abc^2-6b^2c^2$.

(9) Take $10a-b+4c-3d$ from $12a+3b-5c-2d$; and show that the result is numerically correct, when $a=6$, $b=4$, $c=1$, $d=5$.

(10) What number must be added to a to make b; and what number must be taken from $2a^3-6a^2b+6ab^2-2b^3$ to leave $a^3-7a^2b-3b^3$?

MULTIPLICATION.

11.—In multiplying together two simple expressions, four rules must be attended to:

(i) Unlike letters must be written consecutively; $a\times b=ab$.

(ii) Numerical coefficients must be multiplied together arithmetically; $3a\times 4b=12ab$.

(iii) Indices of like letters must be added; $a^3\times a^2=a^5$.

(iv) Like signs will produce $+$, unlike signs $-$, in the result; $a\times b=ab$, $a\times -b=-ab$, $-a\times b=-ab$, $-a\times -b=ab$.

Of these, (i) is the symbolical arrangement already mentioned in (2);

(ii) will be obvious, for $3a$ when multiplied by b will be 3 times as much as a multiplied by b, i. e. $3ab$; and if the multiplier be increased to $4b$, the product will be

4 times $3ab$, i. e. $12ab$.

In (iii), a^3 is $a \times a \times a$, and a^2 is $a \times a$;

$$\therefore a^3 \times a^2 = a \times a \times a \times a \times a = a^5.$$

Expressed generally,

$$a^m \times a^n = a \times a \times a \ldots (m \text{ factors}), \times a \times a \ldots (n \text{ factors})$$
$$= a \times a \times a \times a \ldots (m+n) \text{ factors}$$
$$= a^{m+n}$$

In (iv), it is obvious that a quantity, whether positive or negative, when multiplied by a positive factor, will retain its sign unchanged; thus $+4 \times 3$ means 4 (additive) taken three times, which will be 12, still additive; -4×3 means 4 (subtractive) taken 3 times, which will be 12 subtractive. Multiplication by a *negative* factor combines multiplication and subtraction; thus $+4 \times -3$ means 4 taken three times and made subtractive, i. e. -12; -4×-3 means -12 made subtractive, i. e. $+12$.

Hence follow these rules for the signs:—

+	into	+	produces	+
−	..	+	..	−
+	..	−	..	−
−	..	−	..	+

which are identical with those above stated.

Ex. (1) $2abx^2 \times 6a^2bx^3y^2 = 12a^3b^2x^5y^2$.
Ex. (2) $5m^3x^2y^5 \times -4mnxy^3z = -20m^4nx^3y^8z$.

A compound expression will be multiplied by a simple factor by multiplying its several terms by that factor.

Ex. (3) $(a^2x^3 - 3abx^2y + 4b^2xy^2) \times 2a^2bxy^2 =$
$2a^4bx^4y^2 - 6a^3b^2x^3y^3 + 8a^2b^3x^2y^4$.

Exercise 4.

Multiply—

(1) $3a^2b$ by $2ab^2$; $5a^2x$ by $4ax^2y$; $7a^3bxy$ by b^2xy^3; and $10xy^3z$ by x^4z^4.

(2) $-4ab$ by $3bc$; $2a^3b^2c$ by $-7bc^2$; $-abc$ by $-bc^2d^3$; and $-3a^4bc$ by -6.

(3) 12 by $-7x^3y$; $-4a^3bxy$ by $-abxy^3$; $3x^3y$ by $-4y^3z$; and $-2b^3c^2d$ by $-10a^4bc^3d^4$.

(4) $2a^2-3ab+5b^2$ by $4ab$; $3x^3-2x^2y-7xy^2+y^3$ by $-5x^2y$.

(5) $a^3x-5a^2x^2+ax^3+2x^4$ by ax^2y; $-9a^5+3a^3b^2-4a^2b^3-b^5$ by $-3ab^4$.

When the multiplier is a compound expression, its terms must be used successively as multipliers, and the results added.

Ex. (4) Multiply $2a^2-3ab$ by $3a-4b$

$$\begin{array}{l} 2a^2-3ab \\ \underline{3a\ -4\ b} \\ 6a^3-9a^2b \\ \underline{\qquad\ -8a^2b\ +12ab^2} \\ 6a^3-17a^2b+12ab^2 \quad \textit{Ans.} \end{array}$$

Ex. (5) $(2a-b)\times(a+2c)=2a^2-ab+4ac-2bc.$

Exercise 4—(*continued*).

Multiply—

(6) $2a+b$ by $a+2b$; $2ab-5b^2$ by $3a^2-4ab$; $-a^3+2a^2b-b^3$ by $4a^2+8ab$.

(7) a^2+ab+b^2 by a^2-ab+b^2; $a^3-3a^2b+3ab^2-b^3$ by $a^2-2ab+b^2$.

(8) $x^3-2x^2y+4xy^2-8y^3$ by $x+2y$; $27x^4-18x^2y^2+3y^4$ by $3x^2y+y^3$.

(9) $4x^2yz^2+4xy^2z+y^3$ by $6x^2yz-3xy^2$; $5x^2+y^2$ by $3x^3-5y^3$.

(10) $5a^4-3a^3b+2a^2b^2+ab^3$ by $5a^3b+3a^2b^2-2ab^3+b^4$.

(11) $4a^7y-8a^5y^2+16a^3y^3-32ay^4$ by $a^6y^2+4a^4y^3+4a^2y^4$.

(12) $3m^3+9m^2n+9mn^2+3n^3$ by $2m^4n-6m^3n^2+6m^2n^3-2mn^4$.

(13) $a^2+2ab+b^2$ by $3b-2c$; $4a^3b-2ab^3-b^4$ by $5a^4-6ab^3+b^4$.

(14) $6a^5b+3a^2b^4-2ab^5+b^6$ by $4a^4-2ab^3-3b^4$.

Multiply together—

(15) x^2+xy+y^2, x^2-xy+y^2, and $x^4-x^2y^2+y^4$.

(16) $4a^3-4a^2b+ab^2$, $4a^2+3ab+b^2$, and $2a^2b+b^3$.

(17) $x+a$, $x+2a$, $x-3a$, $x-4a$, and $x+5a$.

(18) $9a^2+b^2$, $27a^3-b^3$, $27a^3+b^3$, and $81a^4-9a^2b^2+b^4$.

(19) From the product of $y^2-2yz-z^2$ and $y^2+2yz-z^2$, take the product of $y^2-yz-2z^2$ and $y^2+yz-2z^2$.

(20) The divisor being $3a^2-ab-3b^2$, the quotient a^2b-2b^3, and the remainder $-2ab^4-6b^5$, find the dividend.

DIVISION.

12.—In dividing one simple expression by another, four rules must be attended to, corresponding to those in multiplication, and derived from them:—

(i) The quotient of two unlike letters must be expressed fractionally; $a \div b = \frac{a}{b}$.

(ii) The quotient of the numerical coefficients must be taken as in arithmetic; $12a \div 3a = 4$.

(iii) The index of a letter in the divisor must be subtracted from that of the like letter in the dividend; $a^5 \div a^3 = a^2$.

If the same power of a letter occur in the divisor and dividend, that letter will disappear from the quotient.

(iv) Like signs will produce +, unlike signs −, in the quotient.

Ex. (1) $32a^5b^3 \div 8a^3b^2 = 4a^2b$.
Ex. (2) $20x^3yz^2 \div -10xz^2 = -2x^2y$.
Ex. (3) $-28ax^3y^2 \div -4bx^2y = \frac{7axy}{b}$.

If the dividend consist of several terms, their quotients must be separately obtained.

Ex. (4) Divide $12x^3y^2 - 16x^2y^3 + 8xy^4$ by $4xy^2$:

$$\begin{array}{r|l} 4xy^2 & 12x^3y^2 - 16x^2y^3 + 8xy^4 \\ \hline & 3x^2 \quad - \quad 4xy \quad + \quad 2y^2. \end{array}$$

Exercise 5.

Divide—

(1) $12a^3b^4$ by $4ab^2$; $10x^3y^2z$ by $2x^2z$; $24a^5xy^4$ by $6axy^4$; $30a^7mx^3$ by $5a^3mx^2$.

(2) $-10x^5y^2$ by $2x^2y$; $28a^3b^3c^3$ by $-7ab^3c^2$; $-18a^3xy^5$ by $-9a^2y$.

(3) $4a^4b - 6a^3b^2 + 12a^2b^3$ by $2a^2b$; $12x^3y^3 - 15x^4y^2 - 24x^5y$ by $-3x^2y$.

(4) $12x^5y - 24x^4y^2 + 36x^3y^3 - 12x^2y^2$ by $12x^2y^2$; $3a^4 - 2a^5b - a^6b^2$ by $-a^4$.

(5) $3x^3yz^2 + 6x^4yz^3 - 15x^5y^2z^3 + 18x^6y^3z - 9x^6y^3z^2$ by $-3x^3yz$.

If the divisor be compound, the same method and arrangement must be employed as in long division in common arithmetic, the first term of the divisor being divided into the first term of each remainder in order to obtain the terms of the quotient.

Ex. (5) Divide $6a^3 - a^2b - 12ab^2$ by $2a - 3b$.

$$
\begin{array}{rll}
2a-3b\,) & 6a^3 - a^2b - 12ab^2 & (\,3a^2 + 4ab \quad \textit{Ans.} \\
 & \underline{6a^3 - 9a^2b} & \\
 & \qquad 8a^2b - 12ab^2 & \\
 & \qquad \underline{8a^2b - 12ab^2} &
\end{array}
$$

Care must be taken in long division to have the divisor and dividend arranged according to the powers of one of their letters, and to have the same arrangement for both.

Ex. (6) Divide $4a^4 + b^4 - 5a^2b^2$ by $b^2 + 2a^2 + 3ab$.

Arranging both by descending powers of a,

$$
\begin{array}{rll}
2a^2+3ab+b^2\,) & 4a^4 - 5a^2b^2 + b^4 & (\,2a^2 - 3ab + b^2 \quad \textit{Ans.} \\
 & \underline{4a^4 + 6a^3b + 2a^2b^2} & \\
 & \quad -6a^3b - 7a^2b^2 + b^4 & \\
 & \quad \underline{-6a^3b - 9a^2b^2 - 3ab^3} & \\
 & \qquad\qquad 2a^2b^2 + 3ab^3 + b^4 & \\
 & \qquad\qquad \underline{2a^2b^2 + 3ab^3 + b^4} &
\end{array}
$$

Exercise 5—(*continued*).

Divide—

(6) $a^2+2ab+b^2$ by $a+b$; $a^2-2ab+b^2$ by $a-b$.

(7) $a^3+2a^2b-3ab^2$ by a^2-ab; $2x^4-5x^3y-3x^2y^2$ by $2x^2+xy$.

(8) $a^3b+a^2b^2-2b^4$ by $a-b$; a^3+b^3 by $a+b$.

(9) $2x^4-7x^3y+2x^2y^2-2xy^3-y^4$ by $2x^2-xy+y^2$.

(10) $16x^4+4x^2y^2+y^4$ by $4x^2-2xy+y^2$.

(11) $32a^5b-56a^4b^2+8a^3b^3-4a^2b^4-ab^5$ by $-4a^2b+6ab^2+b^3$.

(12) $81x^4-y^4$ by $3x-y$; a^4-16b^4 by $a+2b$.

(13) $1+5x^3-6x^4$ by $1-x+3x^2$; $1-51a^3b^3-52a^4b^4$ by $-1+3ab+4a^2b^2$.

(14) x^7y-xy^7 by $x^3y-2x^2y^2+2xy^3-y^4$; x^4+64 by x^2+4x+8.

(15) $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$ by $x^3-3x^2y+3xy^2-y^3$.

(16) $a^7-2a^6b-2a^4b^3+2a^3b^4-6a^2b^5-3ab^6$ by $a^3-2a^2b-ab^2$.

(17) $81x^6y-54x^5y^2-18x^3y^4+18x^2y^5-18xy^6-9y^7$ by $3x^4+x^2y^2+y^4$.

(18) $a^4+2a^3b+8a^2b^2+16ab^3+16b^4$ by a^2+4b^2.

(19) $8y^6-x^6+21x^3y^3-24xy^5$ by $3xy-x^2-y^2$.

(20) $16a^4+9b^4+8a^2b^2$ by $4a^2+3b^2-4ab$.

(21) Find the remainder after dividing 1 by $1+x+x^2$ to six terms in the quotient.

(22) Obtain all the integral terms of the quotient when a^5+1 is divided by a^2-2a+1. What terms must be added to the dividend in order that there may be no remainder?

(23) Divide the product of x^2-y^2 and x^3+y^3 by that of $x-y$ and x^2-xy+y^2.

(24) Divide the sum of $3-5x^2-2x^5$ and $1-4x-7x^2-2x^4$ by their difference.

CHAPTER III.

SIMPLE EXPRESSIONS.

13.—Arithmetical principles respecting factors, fractions, &c., may be readily applied to simple algebraical expressions: and since such expressions are by their very form separated into their prime factors, the application of those principles is in Algebra much easier than in Arithmetic.

14.—Any power of a letter is contained as a factor in any power higher, but in no power lower, than itself: thus a^3 is contained in a^5, but not in a^2.

A simple expression, made up of two or more letters, is contained as a factor in any other simple expression which contains the same letters raised to at least the same powers: thus a^2b^3 is contained in a^3b^3, or in a^2b^4, or in a^5b^6c, &c.

15.—All the factors of a simple expression may be easily written down. For example:

a^2b^3c will have as factors, a, a^2, b, b^2, b^3, c, ab, ab^2, ab^3, ac, a^2b, a^2b^2, a^2b^3, a^2c, bc, b^2c, b^3c, abc, ab^2c, ab^3c, a^2bc, a^2b^2c;

no factor containing a to a power higher than a^2
,, ,, b ,, ,, b^3
,, ,, c ,, ,, c.

The *highest common factor* (or *greatest common measure*) of two or more simple expressions may be written down on inspection. For instance:

$a^4b^3c^2$ and a^2b^4c will have as their highest common factor a^2b^3c.

For it is clear that each will contain a^2, but that the second contains no higher power of a; so b^3 and c are the highest powers of b and c which are contained in both.

Similarly, a^2yz^3, a^4z^2, and a^3yz^2 will have as their highest common factor x^2z^2; y not appearing, since the second expression does not contain it.

Thus it will be seen that the highest common factor of two or more simple expressions will consist of *those letters which are common to all the expressions, each letter being raised to its lowest given power.*

If the expressions have numerical coefficients, the highest common factor of these also must be taken as coefficient for the result. Thus, of $12x^3y$ and $18x^2y^2$, the highest common factor is $6x^2y$.

Note.—Since the highest common factor of two or more expressions contains the highest power of each prime factor which is common to all, it is clear that the highest common factor will contain exactly all other common factors.

For shortness, the highest common factor of expressions is often written as their G. C. M.

16.—The *least common multiple* (which may be written as L. C. M.) of two or more simple expressions may similarly be obtained by inspection. For instance:

a^5x and $a^2x^2y^2$ will have as their least common multiple $a^5x^2y^2$. For it requires

a^5 in order that the	a^5 of the	first	expression may be	contained,	
x^2	„	x^2	second	„	„
y^2	„	y^2	second	„	„

Similarly, that of $a^2bx^2z^4$, $a^3x^3y^3$, and $bx^2y^4z^2$ is $a^3bx^3y^4z^4$.

Thus the least common multiple of two or more simple expressions will consist of *all the letters which appear in them, each letter being raised to its highest given power.*

Numerical coefficients in the expressions will require also their least common multiple to be prefixed as coefficient to the result. Thus the least common multiple of $10a^2b$ and $6bc^2$ is $30a^2bc^2$.

Note.—Since every common multiple of two or more expressions must contain all their factors, raised each to at least its highest power, it will be seen that the least

common multiple will be contained as a factor in every other common multiple of the expressions.

17.—Since the highest common factor of two simple expressions contains their letters raised each to its lower power, while their least common multiple contains the letters raised each to its higher power, it will be seen that *the product of the* G. C. M. *and* L. C. M. *of two expressions is equal to the product of the expressions themselves.*

> Thus the G. C. M. and L. C. M. of $a^2b^3c^4$ and a^3b^5c are a^2b^3c and $a^3b^5c^4$; and in the products it will be seen that the same two powers of each letter are multiplied together—viz. $a^2 \times a^3$, $b^3 \times b^5$, $c^4 \times c$.

18.—The reduction and simplification of fractions will be performed by the same rules as in Arithmetic.

> Ex. (1) $\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}$ will equal $\frac{a^2+b^2+c^2}{abc}$;
> for the least common denominator will be the L. C. M. of bc, ac, ab; and the numerators, found as in Arithmetic, will be a^2, b^2, c^2.
>
> Ex. (2) $\frac{12x^3yz}{15xy^3z}$, when reduced to its lowest terms, will be $\frac{4x^2}{5y^2}$; each part being divided by $3xyz$.

Exercise 6.

(1) Write down all the factors of x^2y^2, and of $10xy$.

(2) Write down all the factors of the third order, of a^3b^2c.

(3) Find the highest common factor of x^3y and x^2y^2; of a^3bc^4 and $a^2b^4c^2$; and of $8xyz^2$ and $12x^3y$.

(4) Find the highest common factor of ab^2c^2, $2a^3bc$, and $3b^2c^3$; and of $20x^3y^2z^4$, $24x^4y^5$, and $16y^5z^4$.

(5) Find the least common multiple of a^3b^2 and ab^4; of $4x^2$ and $6xy^3$; and of $10ax^3y$ and $15bxyz^2$.

(6) Find the least common multiple of $4m^3n$, $10m^2n^3$, and $15lm^4$; and of $5b^3de$, $6a^3b^2c$, $8acd^4$, and $12a^2ce^2$.

(7) Write down the greatest common measure and least common multiple of a^3bx and $a^2x^2y^3$; and of $24x^3y^2$ and $30x^2y^2z$: and illustrate in each case the proposition of (17).

(8) Write down all the factors common to $10a^3b^2c$ and $12a^2c^4$.

(9) The G. C. M. of two expressions is xy^2, and their L. C. M. is x^3y^4z; one of the expressions being x^3y^2, find the other.

(10) The highest common factor of some expressions being a^2bc, write down all the factors of the second order which are common to them.

(11) The highest common factor of some expressions being $6x^2yz^2$, which of the following will be common factors of them, $4xyz$, $2xz^2$, $3y^2z$, x^2z^2?

(12) The least common multiple of some expressions being ab^3c^2, write down all their common multiples of the seventh order.

Simplify—

(13) $\dfrac{16a^3bc}{20a^2bc^2}$. (14) $\dfrac{24xz^5}{32x^4yz^2}$. (15) $\dfrac{18a^4b^3x^2y^4z}{27a^3b^2x^2y^5z^2}$.

(16) $\dfrac{a}{b}$ of $\dfrac{b}{c}$ of $\dfrac{c}{d}$ of $\dfrac{d}{e}$. (17) $\dfrac{xy}{z^2}$ of $\dfrac{yz}{x^2}$ of $\dfrac{xz}{y^2}$.

(18) $\dfrac{15a^2b^3c}{21xy^3z^2}$ of $\dfrac{35x^3y^3}{2b^3c^3}$. (19) $\dfrac{a}{2}+\dfrac{a}{3}+\dfrac{a}{4}$.

(20) $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$. (21) $\dfrac{a^2}{bc}+\dfrac{b^2}{ac}+\dfrac{c^2}{ab}$.

(22) $\dfrac{3x}{2y^2}+\dfrac{5y}{2xz}+\dfrac{1}{z}$. (23) $\dfrac{4xy}{z^2}-\dfrac{xz}{2y^2}$.

(24) $\dfrac{3a^2bc}{5xy^2z}\div\dfrac{9ab^2}{25x^2z}$. (25) $\dfrac{8ab^2}{12bc^2}\times\dfrac{10cd^2}{9a^2b}\div\dfrac{5bcd}{18a^2c}$.

Express as mixed numbers—

(26) $\dfrac{6a^2-12ac+c^2}{6a}$. (27) $\dfrac{x^3y+x^2yz-2z^4}{x^2y}$.

(28) $\dfrac{10a^2bc^2-15ab^2c^2-20a^2b^2c}{5a^2bc}$.

(29) $\dfrac{4a^2xz^2-6xy^3z-9bx^2y^2}{3by^2}$.

Express as single fractions—

(30) $a+\dfrac{ab}{c}+\dfrac{bc}{a}$. (31) $1+\dfrac{x^2y}{z^3}-\dfrac{x^3}{yz^2}$.

(32) $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{ab^2+bc^2+ca^2}{abc}$.

(33) Multiply $\dfrac{12abc}{5xyz}$ by $2a$, by $5z$, and by $15abxy$.

(34) Divide $\dfrac{10x^2y^3z}{9ab^2c^3}$ by $5xz$, by $3abc$, and by $15a^2xy^4$.

(35) Add together $\dfrac{2ax-3by}{4xy}$, $\dfrac{5bx-3az}{6yz}$, and $\dfrac{7bz-5ay}{12xz}$.

(36) From $\dfrac{a^2-2ac+c^2}{a^2c^2}$ take $\dfrac{b^2-2bc+c^2}{b^2c^2}$.

(37) Find the sum of $\dfrac{a}{x}+\dfrac{x}{a}$, $\dfrac{a}{x}\div\dfrac{x}{a}$, and $\dfrac{a}{x}-\dfrac{x}{a}$.

(38) Multiply $1+\dfrac{2ac-3bc}{6ab}+\dfrac{c^2-2ac}{2ac}$ by $\dfrac{6abc-5bc^2}{6abc}-1$.

(39) Simplify $\dfrac{1}{2x^2y}-\dfrac{1}{6y^2z}-\dfrac{1}{2xz^2}+\dfrac{2x-z}{4x^2z^2}+\dfrac{y-2z}{4x^2yz}$.

(40) From $\dfrac{5a^3-2}{8a^2}$ take $\dfrac{3a^2-a}{8}$, and to the result add the sum of the same two fractions.

CHAPTER IV.

SYMBOLICAL EXPRESSION.

19.—RESULTS which arise from the application of Arithmetic to general questions may always be expressed in algebraical symbols, and facility in so expressing them is of great importance.

Ex. (1) The cost of a things at b shillings each will be ab shillings.

Ex. (2) If x miles are travelled uniformly in a hours, the rate is $\frac{x}{a}$ miles per hour.

Such results should be always expressed in one denomination.

Ex. (3) A person who has a shillings, and then buys b things at c pence each, will have left

$$12a-bc \text{ pence,}$$
$$\text{or } a-\frac{bc}{12} \text{ shillings.}$$

Exercise 7.

(1) A man who has £a spends £b; how many pounds will he have left?

(2) At the rate of a miles an hour, how far will be travelled in 4 hours, and in c hours?

(3) How many minutes are there in m hours? and how many in n seconds?

(4) A has x shillings, and B has y shillings; how much will each have when A has given half his money to B?

(5) Of two runners, the faster runs a yards per minute, and the slower b; by what distance will the former win a race which lasts m minutes?

(6) Find the price of a oranges at ten for a shilling, and at b for a shilling.

(7) What rate per minute in yards is the same as a miles per hour?

(8) A pound's worth of articles are bought at x for a shilling, and a pound's worth at y for a shilling, where $x < y$; how many more will there be of one kind than of the other, and what will be the difference between the prices of n of each?

(9) At a shillings for n square feet, what is the price in pence per foot, and per yard in pounds?

(10) Find the circumference and area of a rectangle whose length and breadth are a and b feet.

(11) A room is a feet long, b feet wide, and c feet high; find the area of the walls.

(12) How many yards of carpet x feet wide would be required for the same room? and what would it cost at y pence per yard?

(13) A man who has at first £a pays out b sums of c shillings each, and after receiving d shillings pays away e pence; how many shillings has he left?

(14) A has £x, B has £4 more, C has £3 less than B, and D has as much as B and C together; find D's money.

(15) Write down the number 4 more than twice x, and the number 3 less than four times x.

(16) A number x lies between 20 and 30; subtract its excess above 20 from its defect from 30.

(17) What number exceeds the square of a by 5? what number must be added to this to make it equal to the square of b?

(18) Find in pounds the price of n oranges at a shilling a score.

(19) A regiment of men can be drawn up in a ranks of b men each, and there are c men over; of how many does the regiment consist?

(20) How many men will be required to form a hollow square 4 deep, of which the side contains m men?

(21) From a line whose length is a, are cut off its half, third, and twelfth parts; what length will be left?

(22) A, B, C, D, are four places in a line; the distance AD is a, AC is b, and BD is c; find BC.

(23) A man can complete some work in a hours; what fraction of it will he have done in x hours?

(24) A wine merchant buys n gallons at a shillings per gallon, and sells so as to gain £z on the whole; find the selling price per gallon.

(25) Find the volume in cubic yards of a block a yards long, b feet wide, and c inches thick. If a cubic yard is cut off from the end, what length will be left?

(26) A train which travels at the rate of a miles an hour takes b hours between two stations; what will be the rate of a train which takes c hours?

(27) Find the sum of four successive numbers of which the first is a; and of five, of which the middle one is a.

(28) A person who has d miles to go travels for a hours at b miles per hour, and the rest of the distance at c miles per hour. How long does he take altogether?

(29) Express (in terms of a and b) a number consisting of two digits, the first of which is a and the second b.

(30) A spends c shillings a week, and saves x pounds a year; what will B save in the year, who earns the same wages but spends b shillings a week?

CHAPTER V.

SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

20.—Two algebraical expressions may be so related, that whatever values may be given to the letters involved, they will be equal to one another. Thus for all values of x the two expressions $3x+4x$ and $5x+2x$ are equal.

The statement that the two expressions are equal $\left(\frac{x}{3}+\frac{x}{2}=\frac{3x}{4}+\frac{x}{12}\right)$ is in this case called an *Identity*, or an *Identical Equation*.

But in general two expressions are equal only under certain conditions, i.e. for certain values of the letters involved. Thus, $x+6$ and $4x-3$ are equal if $x=3$, but for no other value: x^2+12 and $9x-8$ are only equal if x has one of the values 4 or 5.

The statement that two such different expressions are equal is called an *Equation*, or sometimes an *Equation of Condition*; as

$$4x-3=x+6 \quad . \quad . \quad . \quad . \quad (1)$$
$$x^2+12=9x-8 \quad . \quad . \quad . \quad . \quad (2)$$

21.—The process of determining the conditions under which the expressions are equal—i.e. of finding the values of the letters which will make them equal—is called *solving* the equation.

Values of the letters which will make the expressions equal are called *Roots* of the equation; and they are said to *satisfy* it. Thus 3 is the root of equation (1) above, and 4 and 5 are the roots of equation (2).

An equation which involves only one letter, whose value is to be determined, is called an equation of *one unknown quantity*; if two, of *two unknown quantities*; &c.

If, when reduced to its simplest form, an equation contains an unknown quantity only to the first power, as in

(1) above, it is said to be of the *first order*, or a *simple* equation; if the unknown quantity occurs in the second power, as in (2) above, it is said to be of the *second order*, or a *quadratic* equation. So equations may be of the *third order*, or *cubic*; of the *fourth order*, or *biquadratic*; or of any higher order.

22.—*Simple Equations of One Unknown Quantity.*—The operation of solving such an equation depends upon these principles:—

(α) *The two sides may have any quantity added to each without destroying their equality.*

(β) *The two sides may be both multiplied or both divided by any number without destroying their equality.*

(α) By the first of these principles, any term of an equation may be removed (or *transposed*) to the other side, provided its sign be changed.

Thus, if $4x-3=x+6$,
then $4x \quad =x+6+3$,
and $4x-x=6+3$, or $3x=9$.

In the first step, 3 being added to each side; in the second, $-x$.

(β) The second of the above principles is applied in particular for two purposes:—

(1) When fractional terms occur in an equation, the whole equation may be multiplied by the least common multiple of the denominators, so as to make all the terms integral.

Thus $\frac{x}{3}-\frac{x}{4}=2-\frac{5x}{6}$ may have its terms multiplied by 12, when it will become

$$4x-3x=24-10x,$$

which is a simpler form.

(2) When the equation has been reduced by transposition to the form obtained in (α), each side may be divided by the coefficient of x.

Thus in the above equation

$$3x=9;$$

dividing by 3, $x=3$.

N.B.—Unknown quantities in equations are generally denoted by the last letters of the alphabet, x, y, z, &c.

Ex. (1) Solve $6+3x=22-5x$

Transposing by (α), so as to have all the terms involving x on one side, and the remaining terms on the other:

$$3x+5x=22-6,$$

i.e. $\quad 8x=16,$

$\therefore$ by (β), $\quad x=2.$ *Ans.*

It will be advisable to verify this result, by noticing that if 2 is written for x in the equation the two sides become equal, each being 12.

Ex. (2) Solve $\frac{2x}{3}+\frac{3x}{5}+\frac{1}{2}=2x-\frac{7x}{10}+\frac{1}{10}$.

Multiplying by 30, the L. C. M. of $3, 5, 2, 10$,

$$20x+18x+15=60x-21x+3,$$

Transposing, $20x+18x-60x+21x=-15+3,$

i.e. $\quad -x=-12,$

or $\quad x=12.$

Ex. (3) Given $\cdot4x-\cdot25x+3=\cdot1x+\cdot25$, find x.

Transposing, $\cdot4x-\cdot25x-\cdot1x=-3+\cdot25,$

i.e. $\quad \cdot05x=-2\cdot75,$

$\therefore \quad x=-55.$

Or, the decimals might have been reduced to fractions, and the equation then solved as in Ex. (2).

Exercise 8.

(1) $5x-1=2x+11.$ (2) $4x+6=x+12.$

(3) $8x-29=26-3x.$ (4) $24x=7x-34.$

(5) $12-5x=5-12x.$ (6) $3x+6-2x=7x.$

(7) $2x-4+3x-6=7x-2+x-5.$

(8) $2x+6+3x=4x-12.$ (9) $15+3x=12x-29-x.$

(10) $x-2-3x+4+5x-6=7x-8+9x.$

(11) $\frac{x}{2} - \frac{x}{3} = 1.$ (12) $\frac{2x}{5} + 2 = \frac{x}{2}.$

(13) $\frac{3x}{4} + \frac{4x}{5} = \frac{5x}{6} + \frac{43}{12}.$ (14) $\frac{7x}{8} = \frac{5x}{12} + 17\frac{1}{2} - x.$

(15) $\frac{24-13x}{5} = \frac{x-12}{3}.$ (16) $\frac{2x}{5} - 3 - \frac{x}{4} = x - \frac{3x}{10} - \frac{1}{4}.$

(17) $\frac{7x}{12} - \frac{3}{4} = \frac{2x}{3} - \frac{3x}{4}.$ (18) $x - \frac{x}{3} - \frac{x}{5} + \frac{x}{6} = \frac{9}{10} - 2\frac{1}{6}.$

(19) $\frac{2x-3}{7} - \frac{1}{2} = \frac{9x}{14} + 3.$ (20) $\cdot 4x + \cdot 6 = \cdot 6x + \cdot 8x - \cdot 3.$

(21) $\cdot 125 - \cdot 3x = \cdot 5x + \cdot 045.$

(22) $\cdot 6x - 1\cdot 4x - 3\cdot 21x = 1 - 4x.$

(23) $\cdot 1x + \cdot 003x - 1 = \cdot 23 - \cdot 02x.$

(24) $\frac{5x}{8} - \frac{x}{6} + 2 = x + \frac{3}{8}.$ (25) $\frac{2x-4}{5} = \frac{x+1}{2} - \frac{5x}{24}.$

(26) $x + \cdot 1 + \cdot 2x = \cdot 3x + 10.$

(27) $4\cdot 173x - \cdot 01 = 1\cdot 013x - \cdot 8.$

(28) $\frac{4x}{9} = x - \frac{3}{2} + \frac{2x}{3} - 4.$ (29) $\frac{23}{12} + \frac{2x}{9} - \frac{x}{6} = \frac{3x}{4} - 1 - \frac{x}{2}.$

(30) $\frac{10x}{11} - \frac{13}{9} + \frac{6x}{7} = \frac{2x}{9} - \frac{4}{11}.$

CHAPTER VI.

PROBLEMS.

23.—Equations are most important in their application to the solution of *Problems.*

Problems so simple as the following may often be solved by ordinary arithmetic, but in general they are too complicated to be brought within its range.

As an example of a problem, and the method of solving it, the following may be taken:

Find a number such that its third and fourth parts together shall exceed its half by 5.

Let x represent the number;

then its half, third, and fourth parts are $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$;

and if x satisfies the conditions of the problem,

$\frac{x}{3}+\frac{x}{4}$ must exceed $\frac{x}{2}$ by 5;

i.e. $\frac{x}{3}+\frac{x}{4}=\frac{x}{2}+5.$

The solution of which equation gives $x=60$. 60 is therefore (as may be verified) the required number.

It will be seen in this example, that the problem and the corresponding equation are mathematically identical, the equation being simply the problem stated symbolically. This symbolical statement is in some cases easily made, but the whole difficulty of a problem usually consists in the separation of its mathematical conditions from extraneous relations, and the translation of them then into symbolical language.

The solution of a problem, it should be remarked, is much facilitated by carefully writing down the symbolical values of all the quantities required before attempting to form the equation.

Ex. (2) Twelve coins, partly shillings and partly half-

crowns, amount altogether to 27*s.*; how many are there of each?

(N.B.—The distinction between *number* of coins and their *value* must be clearly kept in view.)

Let x be the no. of shillings;

then the rest of the 12 (i.e. $12-x$) is the no. of halfcrowns.

The value of the x shillings is $12x$ pence,

and the value of the $12-x$ halfcrowns is $360-30x$ pence.

Now, the conditions of the problem require that these values together shall be 27*s.*, i.e. 324*d.*

$$\therefore\ 360-18x=324$$
$$-18x=324-360$$
$$=-36.$$
$$x=2, \text{ the no. of shillings;}$$
$$\text{and } 12-x=10, \text{ the no. of halfcrowns.}$$

Note.—In forming the equation for such a problem as this, care must be taken to use throughout the same *unit* of value.

In the above case, a penny was taken as the unit; but a sixpence might have been taken; and then

The value of the x shillings would have been $2x$, and of the $12-x$ halfcrowns, $60-5x$; and the equation would have been

$$60-3x=54.$$

Ex. (3) P and Q are two places 20 miles apart; if a person (A) starts from P and walks towards Q at the rate of 3 miles an hour, where will he meet another person (B), who starts 20 minutes later from Q, and walks at the rate of 4 miles an hour?

P———————————————Q

X

Suppose X to be the point at which they meet.

Let x be the number of hours A takes to walk PX;

then B, who starts 20′ later, will take $x-\frac{1}{3}$ hrs. to walk QX.

Hence PX will be $3x$ miles,

and QX will be $4x-\frac{4}{3}$ miles.

But PX+XQ=PQ.

$$\therefore\ 7x-\tfrac{4}{3}=20$$
$$21x-4=60$$
$$21x=64$$
$$x=3\tfrac{1}{21} \text{ hrs.}$$

$\therefore$ A will have walked $9\frac{1}{7}$ miles.

Or, Let $x =$ the distance (PX) which A has walked before meeting B;

then $20-x=$ the distance (QX) which B has walked.

The time which A has taken for walking x miles is $\frac{x}{3}$ hours, and the time which B has taken for $20-x$ miles is $\frac{20-x}{4}$ hours. Also, since B started $20'$ later than A, he has, when they meet, taken $\frac{1}{3}$ hour less than A.

$$\therefore \frac{20-x}{4}=\frac{x}{3}-\frac{1}{3}.$$
$$60-3x=4x-4$$
$$-7x=-64$$
$$x=9\tfrac{1}{7} \text{ miles.}$$

Exercise 9.

(1) Find a number such that when 12 is added to its double the sum shall be 28.

(2) Find the number whose third and fourth parts added together make 14.

(3) What number is that whose third part exceeds its fourth by 14?

(4) The half, fourth, and fifth of a certain number are together equal to 76; find the number.

(5) Find the number whose double exceeds its half by 12.

(6) The combined ages of two children are 15 years, and one is half as old again as the other; what are their ages?

(7) After spending half the money he had in his purse, receiving a sovereign, and then spending five shillings, a person had remaining as much as he had at first; how much had he?

(8) A and B have the same sum; but if A received ten shillings from B, he would then have twice as much as B; how much has each?

(9) In 8 years' time a boy will be three times as old as he was 8 years ago; how old is he?

(10) A sum of £2 is paid in crowns and florins, and the number of crowns is one more than the number of florins; how many are there of each?

(11) A farmer had two flocks of sheep of the same size. He sold out of them 21 and 70 sheep respectively, and then found that he had left in one flock twice as many as in the other; how many had he in each?

(12) From two places 35 miles apart two persons set out at the same time with the intention of meeting, one travelling 6 miles an hour, and the other 8; when and where will they meet?

(13) Divide a distance of 60 yards into three parts, such that the second may be 5 yards more than the first, and the third 10 yards more than the other two together.

(14) A man and a boy earned in a week 37*s.* between them, and if the man had earned 2*s.* more, his wages would have been double the boy's; how much did each earn?

(15) Divide 4*s.* into four parts, such that the first may be less than the second by 2*d.*, greater than the third by 6*d.*, and less than the fourth by 1*s.*

(16) How must £4 10*s.* be divided among four persons, so that the second, third, and fourth may have respectively twice, three times, and four times what the first has?

(17) Find a number whose treble exceeds 50 by as much as its double falls short of 40.

(18) At an election 1533 votes were recorded, and the successful candidate had a majority of 91; how many votes were given for him?

(19) A certain number of sovereigns exceed in value half that number of guineas by £3 16*s.*; find the number of each.

(20) Divide a sovereign between A and B, so that if A give B a fifth of his share they may have equal sums.

(21) A sum of money is to be distributed among a certain number of persons, and it is found that if 5*s.* be given to each there will be a shilling short; if 4*s.* 9*d.*, there will be a shilling over; what is the sum?

(22) Find four consecutive numbers whose sum is 82.

(23) An orchard is planted with cherry, apple, and pear trees. How many are there of each, if it is found that their numbers are respectively 12, 2, and 7 less than half of the whole number of trees?

(24) A traveller sets out from a town, and after going 10 miles a second starts in the same direction, whose rate is $1\frac{1}{4}$ of that of the former; where will he overtake him?

(25) When will the hands of a clock be together between 3 and 4 o'clock?

(26) Two men who can separately complete some work in 15 days and 16 days, can with the help of another do it in 6 days; how long would the third man take by himself?

(27) A number consisting of two digits, of which that in the unit's place is 7, exceeds by 18 the number formed by reversing the order of the digits; find the number.

(28) A fruiterer sold some pears at 2 a penny, and the same number at 3 a penny; if he had mixed them, and sold them at the rate of 5 for 2*d.*, he would have received 3*d.* less; how many did he sell?

(29) Twenty-four coins, consisting of shillings and half-crowns, amount altogether to £1 19*s.*; how many half-crowns are there?

(30) A boy who runs at the rate of 12 yards per second, starts 20 yards behind another whose rate is $10\frac{1}{2}$ yards per second; how soon will he be 10 yards before him?

CHAPTER VII.

COMPOUND TERMS AND COEFFICIENTS.

24.—The terms of an expression have been hitherto considered to be *simple*; but they may be also *compound*, i.e. made up of several simple ones, which will then be enclosed within brackets.

Thus the expression $(3x-4y)+(2x+y)$ may be considered to consist of the two compound terms, $3x-4y$ and $2x+y$.

A compound term may have a coefficient, which will be written outside its bracket; thus in $3(ax-by)-2(bx+3cy)$, the first term is *three times* $ax-by$, and the second *twice* $bx+3cy$.

Similarly, *coefficients* may be compound; thus, in $(2a+b)x+(a-3b)y$, the coefficient of x is $2a+b$, and of y, $a-3b$.

Compound terms, or coefficients, become sometimes more complex in their forms, by themselves comprising other compound terms; as in

$$\{a-(2b+3c)\}-\{4(a-c)-2(b-5c)\}$$

in which the first compound term is made up of the simple term a, and the compound term $2b+3c$; and the second of the two compound terms, $4(a-c)$ and $2(b-5c)$.

25.—An expression may have its compound terms readily resolved into simple terms, as in the following examples:—

Ex. (1) $\quad (3x-4y)+(2x-y)=3x-4y+2x-y$
$$=5x-5y.$$

Ex. (2) $\quad (3x-4y)-(2x-y)=3x-4y-2x+y$
$$=x-3y.$$

In the first of these examples, $2x-y$ is to be added to $3x-4y$, which will be done by adding their separate terms (9, page 8); but in the second, $2x-y$ is to be subtracted from $3x-4y$, i.e. its terms must have their signs changed, and then be added.

The resolution of the compound terms of an expression into simple ones consists, therefore, in the removal of its brackets; and the general rule will be, that *a bracket having + before it may be removed without altering the signs within, but if a bracket having − before it be removed, the signs of all the terms within it must be changed.*

The coefficients of a compound term must be multiplied into each of its simple terms; thus:

Ex. (3)
$$\begin{aligned}&3(ab-ac)-2(ab+4bc)+3(ac-3bc)\\&=3ab-3ac-2ab-8bc+3ac-9bc\\&=ab-17bc.\end{aligned}$$

When the compound terms of an expression themselves comprise compound terms—i.e. when pairs of brackets enclose other brackets—the larger brackets should be first removed, and then the smaller.

Ex. (4)
$$\begin{aligned}&\{2x-(3y-z)\}-\{(3x+2y)-z\}\\&=2x-(3y-z)-(3x+2y)+z\\&=2x-3y+z-3x-2y+z\\&=-x-5y+2z.\end{aligned}$$

Ex. (5)
$$\begin{aligned}&[5a+4(b-2c)]-4[a+2(b-c)]\\&=5a+4(b-2c)-4a-8(b-c)\\&=5a+4b-8c-4a-8b+8c\\&=a-4b.\end{aligned}$$

Besides the ordinary forms [ ], { }, ( ), being used for brackets, a bar is sometimes written over two or more terms for the same purpose: thus, $a-\overline{b-2c}$ is the same as $a-(b-2c)$.

26.—Compound coefficients will be resolved into simple ones in the same manner.

Ex. (6)
$$\begin{aligned}&(a-b)x-(2b-c)y+3(c-a)z\\&=ax-bx-2by+cy+3cz-3az.\end{aligned}$$

A compound term may have a compound coefficient; in which case they may either be separately resolved, or their product may be written as a compound term, and then resolved into simple terms.

Ex. (1) $(a+b)(x-2y)-2(a-b)(x+y)$

$$\begin{aligned}&=a(x-2y)+b(x-2y)-2a(x+y)+2b(x+y)\\&=ax-2ay+bx-2by-2ax-2ay+2bx+2by\\&=-ax+3bx-4ay.\end{aligned}$$

Or, $(a+b)(x-2y)-2(a-b)(x+y)$

$$\begin{aligned}&=(ax+bx-2ay-2by)-2(ax-bx+ay-by)\\&=ax+bx-2ay-2by-2ax+2bx-2ay+2by\\&=-ax+3bx-4ay.\end{aligned}$$

It will be observed in all cases that a compound term, or a compound coefficient, is to be regarded *as a whole*, affected throughout by any sign or coefficient placed before it, and in this differing from a mere collection of simple terms, each of which is affected only by its own sign and coefficient.

The index of a compound term is written outside the bracket; as $(a+b)^3$.

Exercise 10.

Express with simple terms—

(1) $(a+b)+(b+c)-(a+c)$.

(2) $(2a-b-c)-(a-2b+c)$.

(3) $(2x-y)-(2y-z)-(2z-x)$.

(4) $(a-x-y)-(b-x+y)+(c+2y)$.

(5) $(2x-y+3z)+(-x-y-4z)-(3x-2y-z)$.

(6) $(3a-b+7c)-(2a+3b)-(5b-4c)+(3c-a)$.

(7) $\{a-(b-c)\}+\{b-(c-a)\}-\{c-(a-b)\}$.

(8) $\{2x+(y-3z)\}-\{(3x-2y)+z\}+5x-(4y-3z)$.

(9) $\{(3a-2b)+(4c-a)\}-\{a-(2b-3a)-c\}+\{a-(b-5c-a)\}$.

(10) $\{a-(2a+\overline{3a-4a})\}-5a-\{6a-(\overline{7a+8a}-9a)\}$.

(11) $2(a+b)-3(c-a)+4(b-c)$.

(12) $5(a-b+c)-2(a+3c)+3(a-2b)$.

(13) $(a-3)x-(2b-1)x-(a-2b)x$.

(14) $6(a+b)x-4(a-b)y-2a(3x-2y)$.

(15) $\{3a-2(b-c)\}-\{4b-3(c-a)\}+6(b-c)$.

(16) $5\{x^2-2(x+1)\}-7\{x^2+(3x+2)\}-12x-2\{-x^2-3(3x+4)\}$.

(17) $a-\{2(x-y)-3b\}-\{3(b+3y)-7(2x+y)\}+4(a-3x)$.

(18) $2[a-2\{a-2(a-2\overline{a-2a})\}]-3(a-3\overline{a-3a})$.
(19) $(a+b)(x+y)-(a-b)(x-y)$.
(20) $(a-2b)(x+y)+(x-2y)(a-b)$.
(21) $2(m-3n)(3x-y)-3(2m+n)(x+2y)$.
(22) $\{ac-(a-b)(b+c)\}-b\{b-(a-c)\}$.
(23) $5\{(a-b)x-cy\}-2\{a(x-y)-bx\}-\{3ax-(5c-2a)y\}$.
(24) $(x-1)(x-2)-3x(x+3)+2\{(x+2)(x+1)-3\}$.

27.—Conversely, two or more simple terms of an expression may be combined into one compound term. If the sign + be placed before it, each of its simple terms will retain its own sign; if −, each of its simple terms must have its sign changed, as will be obvious from (25). It is usual to give the compound term the sign of its first simple term.

Ex. (1)

$a-b+c-d-e+f$, if expressed in three binomial terms (i.e. terms containing two simple terms), will be—

$$(a-b)+(c-d)-(e-f) \quad . \quad . \quad . \quad (\alpha)$$

If expressed in two trinomial terms, it will be—

$$(a-b+c)-(d+e-f) \quad . \quad . \quad . \quad . \quad (\beta)$$

This last form may have the last two terms in each bracket enclosed within another bracket—

$$\{a-(b-c)\}-\{d+(e-f)\} \quad . \quad . \quad (\gamma)$$

The first expression may, in the same manner, have others of its simple terms combined into compound terms. Thus if the last five terms are enclosed within brackets, then the last four within another pair of brackets, and so on, it will become—

$$a-[b-\{c-(d+\overline{e-f})\}] \quad . \quad . \quad . \quad (\delta)$$

28.—A simple coefficient common to all the terms which are enclosed within a pair of brackets may be written outside the brackets, as a coefficient to the whole compound term.

Ex. (2)

$5a-10b-6x+12y$, expressed in binomial terms, will be—

$$(5a-10b)-(6x-12y)$$
$$=5(a-2b)-6(x-2y).$$

the same manner as above, a letter occurring in an ssion with several simple coefficients may be written ıd with one compound coefficient.

(3) $ax+2bx+cx^2-3dx^2=(a+2b)x+(c-3d)x^2.$

(4) $4ax+3by-15cz-8bx+6cy-10az$

$$=(4a-8b)x+(3b+6c)y-(15c+10a)z$$

or, $4(a-2b)x+3(b+2c)y-5(3c+2a)z.$

Exercise 11.

press in (i) binomial, (ii) trinomial terms:

$2a-3b-4c+d+3e-2f.$ (2) $a-2x+4y-3z-2b+c.$

$a^5+3a^4-2a^3-4a^2+a-1.$

$-3a-2b+2c-5d-e-2f.$

$ax-by-cz-bx+cy+az.$

$2x^5-3x^4y+4x^3y^2-5x^2y^3+xy^4-2y^5.$

Express each of the above in trinomial terms hav- heir last two simple terms enclosed within inner ets.

ite in binomial terms (with simple factors outside rackets):—

$3a-6b-4c+8d.$ (9) $ax+ay-bx+by.$

$2ax-6ay+4bz-4bx-2cx-3cy.$

$5x^2-10xy-3y^2+9yz+xz+2z^2.$

lect in brackets the coefficients of x, y, z, in—

$ax-bx+2ay+3y+4az-3bz-2z.$

$ax-2by+5cz-4bx-3cy+az-2cx-ay+4bz.$

$a(x-y)+b(y-z)-c(z-x).$

$12a(x+y)+4b(y-3z)-3c(5x-2\overline{y-z}).$

The sum of $ax-by+cz$, $2bx-ay-3cz$, and $by-az.$

The sum of $(a+b)x-(a+c)y+(b+c)z$, and $a(2x+y)-b(x-z)-c(y-2z).$

$\{2ax-3by-7cz\}-2\{(b-c)x-4cz\}-c\{2x+(y+z)\}.$

Multiply together $x-a$, $x+b$, and $x-c$; collecting ıckets the coefficients of the powers of x.

Simplify $(a-b)(x+z)-3(b+c)(y-x)-$ $a)(z+y)$, expressing the result with compound co- nts for x, y, z.

CHAPTER VIII.

COMPOUND FACTORS.

29.—**Algebraical work** will be often much simplified by facility in dealing with binomial factors.

For this purpose three important results in multiplication must first be noticed and remembered:

(i) $a+b$ multiplied by itself gives

$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad\;\; ab+b^2} \\ a^2+2ab+b^2; \end{array}$$

i.e. *the square of the sum of two terms is equal to the sum of their squares + twice their product.*

(ii) $a-b$ multiplied by itself gives

$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\quad -ab+b^2} \\ a^2-2ab+b^2; \end{array}$$

i.e. *the square of the difference of two terms is equal to the sum of their squares — twice their product.*

(iii) $a+b$ multiplied by $a-b$ gives

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2 \qquad -b^2; \end{array}$$

i.e. *the product of the sum and difference of two terms is equal to the difference of their squares.*

By remembering these, the square of any binomial expression, or the product of the sum and difference of any two terms, may be written down by inspection.

Ex. (1) $(3x+2y)^2 = 9x^2+12xy+4y^2$.
Ex. (2) $(2a^2x-5x^2y)^2=4a^4x^2-20a^2x^3y+25x^4y^2$.
Ex. (3) $(3ab^2c+2a^2c^2)(3ab^2c-2a^2c^2)=9a^2b^4c^2-4a^4c^4$.

Note.—If two expressions differ only in some of their signs, it is often convenient to write them in one formula, with double signs between the terms. Thus, $a\pm b$ includes both $a+b$ and $a-b$. If two such expressions are written equal to one another, the upper signs are to be understood to correspond with one another, and the lower with one another.

Thus $(a\pm b)^2=a^2\pm 2ab+b^2$ represents both
$(a+b)^2=a^2+2ab+b^2$
and $(a-b)^2=a^2-2ab+b^2$.

Exercise 12.

Write down the squares of—
(1) $x+y$, $y-z$, $2x+1$, $2a+5b$, $1-x^2$.
(2) $3ax-4x^2$, $1-7a$, $5xy+2$, $ab+cd$, $3mn-4$.
(3) $12+5x$, $4xy^2-yz^2$, $3abc-bcd$, $4x^3-xy^2$.

Write down the products of—
(4) $x+y$ and $x-y$; $2a+b$ and $2a-b$; $3-x$ and $3+x$.
(5) $3ab+2b^2$ and $3ab-2b^2$; $4x^2-3y^2$ and $4x^2+3y^2$; $a^3x^2-by^4$ and $a^3x^2+by^4$.
(6) $6xy-5y^2$ and $6xy+5y^2$; $4x^5-1$ and $4x^5+1$; $1+3ab^3$ and $1-3ab^3$.

(7) Write down the square of the product of $a+2x$ and $a-2x$; and of $2bx-3xy$ and $2bx+3xy$.
(8) Find the value of $(a-x)(a+x)(a^2+x^2)(a^4+x^4)$.

Simplify—
(9) $(a+b)^2+(a-b)^2$. (10) $(x+y)^2-(x-y)^2$.
(11) $2(a-2b)^2-4(a+b)^2$.
(12) $(a+b)(a-b)+(b+c)(b-c)+(c+a)(c-a)$.
(13) $(a+2b)^2-4(a+3b)(a-3b)+4(a-2b)^2$.
(14) $x(2x-y)^2-3x^2(x+2y)+5(x^2-y^2)y$.
(15) $\{(2a+b)^2+(a-2b)^2\}\times\{(3a-2b)^2-(2a-3b)^2\}$.

(16) $4(a-3b)(a+3b)-2(a-6b)^2-2(a^2+6b^2)$.
(17) $x^2(x^2+y^2)^2-2x^2y^2(x+y)(x-y)-(x^3-y^3)^2$.
(18) $16(a^2+b^2)(a^2-b^2)-(2a-3)(2a+3)(4a^2+9)+(2b-3)(2b+3)(4b^2+9)$.
(19) Multiply the excess of the square of a over the square of $a-b$ by its defect from the square of $a+b$.
(20) Express symbolically, and simplify—
The excess of the sum of the squares of the sum and difference of a and b over twice the product of that sum and difference.

30.—Conversely, a trinomial expression, consisting of the sum of the squares of two terms and twice their product (i.e. of the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$), may be recognised as a perfect square; and its root will consist of those two terms connected by the sign which affects the double product.

Ex. (1) $4a^2+12ab^2+9b^4$ is the square of $2a+3b$.
Ex. (2) $9x^2y^4-30xy^2z+25z^2$ is the square of $3xy^2-5z$.

Also, since the product of the sum and difference of two terms is the difference of their squares, it follows that the difference of the squares of two terms may always be resolved into two factors—viz. the sum and difference of those terms.

Ex. (3) $4a^2x^2-b^2y^2=(2ax+by)(2ax-by)$.
Ex. (4) $3x^5-12xy^4=3x(x^4-4y^4)=3x(x^2+2y^2)(x^2-2y^2)$.

Exercise 13.

(1) Of what expressions are $a^2+2abc+b^2c^2$, $9x^2-6xy+y^2$ $16a^4-24a^2x^3+9x^6$, and $1+4y^2+4y^4$ respectively the squares?
(2) Of what are $25-40x+16x^2$, $a^4b^2+4a^2b^4c^2+4b^6c^4$, and $49x^6-42x^3yz^2+9y^2z^4$, the squares?
(3) Which of the expressions $a^2-4ab+b^2$, $4a^2-4ab+b^2$, $4a^2-4ab-b^2$, $4x^2-20xy+25y^2$, $x^4+x^2y^2+y^4$, are perfect squares?
(4) What must be inserted as the middle terms in the

expressions a^2+x^2, $4a^2+y^2$, $a^4+25a^2b^2$, $9x^6+4y^4$, in order to make them perfect squares?

(5) What terms must be affixed to x^2+2ax, x^2-4ax, $4y^2-4yz$, y^2-y, x^2+bx, in order to make them complete squares?

(6) Find three terms either of which added to $x^4+4b^2x^2$ will make a complete square.

Resolve into factors—

(7) x^2-36y^2, $4a^2-z^2$, $18y^2-50z^2$, $16y^3-a^2b^2y$.

(8) $25a^2b^4-4b^2c^4$, $36a^4x^3y^2-25b^2x^5y^2$, $100-x^2y^2$.

(9) x^2-1, a^4-4a^2, $1-9x^2z^4$, $9a^3x^8-16a$.

(10) Resolve into their simplest factors a^4-b^4, $16x^4-1$, a^8-1, x^5y-xy^5.

(11) Write down the quotient of a^2-b^2 by $a-b$; of x^2-1 by $x+1$; of $a^2x^2-4b^2y^2$ by $ax+2by$; of $64a^4-25b^4$ by $8a^2-5b^2$.

(12) Write down the quotient of $4x^2-4xy+y^2$ by $2x-y$; of $9a^2x^2+6ax+1$ by $3ax+1$; of $16a^4-40a^2xy+25x^2y^2$ by $4a^2-5xy$.

31.—Again, the product of two binomial factors of the form $x+a$, $x+b$, should be noticed:

(i)
$$\begin{array}{r} x+5 \\ x+3 \\ \hline x^2+5x \quad\quad \\ 3x+15 \\ \hline x^2+8x+15 \end{array}$$

(ii)
$$\begin{array}{r} x-5 \\ x-3 \\ \hline x^2-5x \quad\quad \\ -3x+15 \\ \hline x^2-8x+15 \end{array}$$

(iii)
$$\begin{array}{r} x+5 \\ x-3 \\ \hline x^2+5x \quad\quad \\ -3x-15 \\ \hline x^2+2x-15 \end{array}$$

(iv)
$$\begin{array}{r} x-5 \\ x+3 \\ \hline x^2-5x \quad\quad \\ +3x-15 \\ \hline x^2-2x-15 \end{array}$$

On examining these products, the following points will be observed:—

(*a*) In all, the first term is x^2, and the last is the product of the 5 and 3.

(β) (i) and (ii) show that when the second terms of the factors have both the *same* sign, the product has

(1) its last term positive;

(2) the coefficient of its middle term = sum of 3 and 5;

(3) the sign of its middle term the same as that of the 3 and 5.

(γ) (iii) and (iv) show that when the second terms of the factors have *opposite* signs, the product has

(1) its last term negative;

(2) the coefficient of its middlo term = difference of 3 and 5;

(3) the sign of its middle term is that of the 5, the greater of the two numbers.

Note.—These results may be deduced from the one general formula $(x+a).(x+b)=x^2+(a+b)x+ab$

by supposing for (i), a and b both positive;
(ii), a and b both negative;
(iii), a positive, b negative, and $a > b$;
(iv), a negative, b positive, and $a > b$;

Exercise 14.

Write down the products of—

(1) $x+2$ and $x+3$; $x+1$ and $x+5$; $x-3$ and $x-6$.
(2) $x-9$ and $x-1$; $x-8$ and $x+1$; $x-2$ and $x+5$.
(3) $x-3$ and $x+7$; $x-2$ and $x-4$; $x+1$ and $x+11$.
(4) $x-2a$ and $x+3a$; $x-c$ and $x-d$; $x-4y$ and $x+y$.
(5) $a-2b$ and $a-5b$; x^2+2y^2 and x^2+y^2; x^2-3xy and x^2+xy.

32.—Conversely, from the above results may be deduced a method for resolving into its factors, if any, a trinomial of the form x^2+ax+b:

(i) If the last term be *positive*, find two numbers whose product is that last term, and whose *sum* is the coefficient of the middle term: those two numbers will be the second terms of the factors, and each will have the sign of the given middle term.

(ii) If the last term be *negative*, find two numbers whose product is that last term, and whose *difference* is the coefficient of the middle term: those two numbers will be the second terms of the factors; the greater of the two will have the sign of the given middle term, and the less will have the opposite sign.

Ex. (1) Resolve into its factors $x^2-7x+10$.

Here the last term is positive;

also $10=5\times 2$, $7=5+2$:

$\therefore$ the middle term being negative,

$$x^2-7x+10=(x-5)(x-2).$$

Ex. (2) Resolve $x^2+5x-14$.

Here the last term is negative;

also $14=7\times 2$, $5=7-2$:

and 7 will have the sign + of the middle term;

$$\therefore\ x^2+5x-14=(x+7)(x-2).$$

Simple factors should be first separated.

Ex. (3) $3x^2-24x-60=3\,(x^2-8x-20)$

$\qquad\qquad\qquad\qquad\quad =3\,(x-10)\,(x+2)$.

Exercise 15.

Resolve into factors—

(1) $x^2+7x+12$; x^2-7x+6; $x^2+11x+24$; x^2+3x+2.
(2) x^2-5x+4; x^2+2x-3; $y^2-3y-10$; y^2-y-6.
(3) x^2+7x+6; x^2+6x-7; $x^2-5xy+4y^2$; $a^2-3ab+2b^2$.
(4) $5x^2-15x-20$; $3y^2+6y-105$; $m^2-5am-50a^2$.
(5) $x^4-4a^2x^2+3a^4$; $2x^5-16x^4+24x^3$; $a^2x^2-3ax-54$.
(6) Obtain without division the quotient of $x^2-12x+27$ by $x-3$; and of $3a^2b^2-9ab-12$ by $ab+1$.

33.—It may be observed that (29) and (30) are but particular cases of (31) and (32); for the general formula

$$(x+a)(x+b)=x^2+(a+b)x+ab$$

becomes, when $b=a$,

$$(x+a)(x+a),\text{ i. e. }(x+a)^2=x^2+2ax+a^2.$$

Or, if a be negative,

$$(x-a)\,(x-a),\text{ i. e. }(x-a)^2=x^2-2ax+a^2;$$

and when $b=-a$, it becomes

$$(x+a)(x-a)=x^2-a^2.$$

So, conversely, the resolution of $x^2+2ax+a^2$ and $x^2-2ax+a^2$, depends upon finding two numbers whose product is a^2, and whose sum is $2a$; and that of x^2-a^2, in which the coefficient of the middle term is 0, upon finding two numbers whose product is a^2, and whose difference is 0, i. e. a and a.

34.—The above methods of writing down products, and of resolving expressions into factors, may be applied to binomial expressions which consist of compound as well as simple terms.

Expressions may be written as binomials (27), in order to make the methods applicable.

Ex. (1) The product of $(x+y)+a$ and $(x+y)-a$ is

$$(x+y)^2-a^2, \text{ or } x^2+2xy+y^2-a^2.$$

Ex. (2) $(2x^2-3x+1).(2x^2+3x-1)=$

$$\begin{aligned}&\{2x^2-(3x-1)\}.\{2x^2+(3x-1)\}\\&=4x^4-(3x-1)^2\\&=4x^4-9x^2+6x-1.\end{aligned}$$

Ex. (3)
$$\begin{aligned}a^2-b^2-2bc-c^2&=a^2-(b^2+2bc+c^2)\\&=a^2-(b+c)^2\\&=\{a+(b+c)\}\{a-(b+c)\}\\&=(a+b+c)(a-b-c).\end{aligned}$$

Exercise 16.

Write down the products of—

(1) $(a+x)+3$ and $(a+x)+1$; $x+(a-5)$ and $x-(a-1)$.

(2) $(a-b)+c$ and $(a-b)-c$; $x^2+(x-1)$ and $x^2-(x-1)$.

(3) $(a-b)-(c-d)$ and $(a-b)+(c-d)$; $x-y-z$ and $x-y+z$.

(4) $2+x+x^2$ and $2-x-x^2$; a^2-ab+b^2 and a^2+ab+b^2.

Resolve into factors—

(5) $(a-b)^2-x^2$; $(x+2y)^2-4z^2$; $(a-b)^2-(c+d)^2$.

(6) $(x+1)^2-(y+1)^2$; $(x+1)^2-(y-1)^2$; $d^2-(x-2y)$

(7) $a^2-2ab+b^2-c^2$; $a^2-b^2+2bc-c^2$; a^2-b^2-4b-4.

(8) $x^4+x^2y^2+y^4$; $x^4-3x^2y^2+y^4$; a^4+a^2+1.

(9) Write down the squares of $a+b+c$; $x-y-z$; x^2-2x+3; $2x^4+x-1$.

(10) Prove that

$$\begin{aligned}(a+b+c+d+\ldots)^2 = {} & a^2+2a(b+c+d+\ldots)\\ & +b^2+2b(c+d+\ldots)\\ & +c^2+2c(d+e+\ldots)\\ & +\ \&c.\end{aligned}$$

(11) Resolve into factors $(a+x)^2-3(a+x)+2$; $a^2-ab-ac+bc$.

(12) Resolve $x^2+2xy+y^2-x-y-6$; $x^4+2x^3+5x^2+4x+3$.

Prove that—

(13) $(a+b-c-d)^2-(a-b+c-d)^2=4(a-d)(b-c)$.

(14) $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2 = 4ab(a^2+b^2)$.

(15) $(a+b+c)^2+(a+b-c)^2+(a-b+c)^2+(b+c-a)^2$
$=4(a^2+b^2+c^2)$.

(16) $(a+b)^2+2(a^2-b^2)+(a-b)^2=(2a)^2$.

(17) $\{(a+1)b-(a-1)\}^2-\{(b+1)a-(b-1)\}^2$
$=4(ab+1)(b-a)$.

(18) $(a+b+c)(a+b-c)(a-b-c)(a-b+c)$
$=a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2)$.

(19) $\left(1+\dfrac{a^2+b^2-c^2}{2ab}\right)\left(1+\dfrac{a^2+b^2+c^2}{2ab}\right)=\dfrac{(a+b)^4-c^4}{4a^2b^2}$.

(20) $\left(\dfrac{a}{b}+1+\dfrac{b}{a}\right)\cdot\left(\dfrac{a}{b}-1+\dfrac{b}{a}\right)\cdot\left(\dfrac{a^2}{b^2}-1+\dfrac{b^2}{a^2}\right)=\dfrac{a^4}{b^4}+1+\dfrac{b^4}{a^4}$.

35.—The terms of an expression may sometimes be so arranged as to show a compound factor running through the whole:

Ex. (1) $ax-ay+bx-by=a(x-y)+b(x-y)$
$=(a+b)(x-y)$.

Ex. (2) $x^2-y^2-xz+yz=(x+y)(x-y)-(x-y)z$
$=(x+y-z)(x-y)$.

When an expression consists of sets of terms of different orders, the resolution will often be facilitated by observing the factors of which the set of highest order is composed.

Ex. (3) $x^2-3xy+2y^2-3x+6y$
$=(x-2y)(x-y)-3(x-2y)$
$=(x-2y)(x-y-3)$.

Exercise 17.

Resolve into factors—

(1) $xy-xz+by-bz$; $ab-ac-b^2+bc$.

(2) $x^2-xy+xz-yz$; $3x^2-xy-3xz+yz$.

(3) $a^2-x^2-ab-bx$; $a^2-2ax+x^2+a-x$.

(4) $3x^2-3y^2-2x+2y$; $m^2p-m^2q-n^2p+n^2q$.

(5) $x^4+x^3+x^2+x$; $a^4x^4-a^3x^3-a^2x^2+1$.

(6) $3x^3-2x^2y-27xy^2+18y^3$; $4x^4-x^2+2x-1$.

(7) $1-x+x^2-x^3$; $1-x-x^2+x^3$.

(8) $(x+y)^2-xy(x+y+1)-1$.

(9) $(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2$.

(10) $x^2-y^2-z^2+2yz+x+y-z$.

(11) $a^2-ab-6b^2-4a+12b$.

(12) $x^3-2x^2y+x^2-4x+8y-4$.

36.—It will be observed that compound expressions may contain simple factors; and that when such is the case, these will be at once determinable on inspection. But a compound expression of the form hitherto considered cannot, when multiplied by another expression, have a simple product; consequently a simple expression cannot contain a compound factor.

CHAPTER IX.

GREATEST COMMON MEASURE OF COMPOUND EXPRESSIONS.

37.—The Greatest Common Measure of two or more compound expressions is *the factor of highest dimensions which is contained in each.*

Two expressions which contain no common factor except 1 are said to be *prime* to one another.

When compound expressions can be separately resolved into their factors, their G. C. M. may be obtained by inspection, in the same manner as that of simple expressions. It will consist of all the factors common to the expressions, each factor being written with its lowest index.

Ex. (1) Find the G. C. M. of $2a^2x+2ax^2$ and $3abxy+3bx^2y$.

$$2a^2x+2ax^2=2ax(a+x)$$
$$3abxy+3bx^2y=3bxy(a+x)$$
$$\therefore \text{G. C. M.}=x(a+x).$$

Ex. (2) Find the G. C. M. of $8a^2x^2-24a^2x+16a^2$ and $12ax^2y-12axy-24ay$.

$$8a^2x^2-24a^2x+16a^2=8a^2(x^2-3x+2)$$
$$=8a^2(x-1)(x-2)$$
$$12ax^2y-12axy-24ay=12ay(x^2-x-2)$$
$$=12ay(x+1)(x-2)$$
$$\therefore \text{G. C. M.}=4a(x-2).$$

Exercise 18.

Find the G. C. M. of

(1) $7x^2-4x$, $7a^2x-4a^2$.

(2) $12a^3x^2y-4a^3xy^2$, $30a^2x^3y^2-10a^2x^2y^3$.

(3) $8a^3b^2c - 12a^2bc^3$, $6ab^4c + 4ab^3c^2$.
(4) $x^2 - 2x - 3$, $x^2 + x - 12$.
(5) $2a(a^2 - b^2)$, $4b(a+b)^2$.
(6) $12x^3y(x-y)(x-3y)$, $18x^2(x-y)(3x-y)$.
(7) $3x^3 + 6x^2 - 24x$, $6x^3 - 96x$.
(8) $ac(a-b)(a-c)$, $bc(b-a)(b-c)$.
(9) $10x^3y - 60x^2y^2 + 50xy^3$, $5x^2y^2 - 5xy^3 - 100y^4$.
(10) $x(x+1)^2$, $x^2(x^2-1)$, $2x(x^2-x-2)$.
(11) $3x^2 - 6x + 3$, $6x^2 + 6x - 12$, $12x^2 - 12$.
(12) $6(a-b)^4$, $8(a^2-b^2)^2$, $10(a^4-b^4)$.

38.—When the G. C. M. of two compound quantities is required which are not resoluble by inspection, the method to be employed is similar to that adopted in the corresponding case in Arithmetic. And just as that consists in obtaining a series of continually decreasing numbers, which we know to contain as a factor the G. C. M. required; so in Algebra we shall obtain a series of expressions of continually decreasing dimensions, each of which will contain as a factor the required G. C. M.

The method depends upon two principles:

(α) *Any factor of an expression will be a factor also of any multiple of that expression.*

For if F represent a factor of an expression A, so that $A = nF$; then $mA = mnF$, and therefore also contains F as a factor.

(β) *Any common factor of two expressions will be a factor of their sum or difference, or of the sum or difference of any multiples of them.*

For if A and B are two expressions containing a common factor F, so that

$$A = mF,\ B = nF;$$

then

$$A \pm B = mF \pm nF = (m \pm n)F.$$

Also

$$pA \pm qB = pmF \pm qnF = (pm \pm qn)F.$$

Ex. (1) Find the G. C. M. of $2x^2+x-3$ and $4x^3+8x^2-x-6$

$$
\begin{array}{rl}
2x^2+x-3\,)\,4x^3+8x^2-x-6\,(\,2x+3 & \\
\underline{4x^3+2x^2-6x} & \\
6x^2+5x-6 & \\
\underline{6x^2+3x-9} & \\
2x+3\,)\,2x^2+x-3\,(\,x-1 & \\
\underline{2x^2+3x} & \\
-2x-3 & \\
\underline{-2x-3} &
\end{array}
$$

and G. C. M. $=2x+3$.

As in Division (12), care must be taken to have both the given expressions arranged by ascending or descending powers, and the latter is generally preferable.

When arranged by descending powers, the expression whose first term is of the lower order must, of course, be taken for divisor; or if the first terms are of the same order, that which has the smaller coefficient: and each division is to be continued until the remainder is of lower dimensions than the divisor.

This method is of use only to determine the *compound* factor of the G. C. M. Simple factors of the given expressions must be separated from them before applying it; and the greatest common measure of these must be observed and multiplied into the compound factor obtained.

Ex. (2) Find the G. C. M. of $12x^4+30x^3-72x^2$ and $32x^3+84x^2-176x$.

$$12x^4+30x^3-72x^2=6x^2(2x^2+5x-12),$$
$$32x^3+84x^2-176x=4x\,(8x^2+21x-44),$$

and $6x^2$, $4x$ have $2x$ common.

$$
\begin{array}{rl}
2x^2+5x-12\,)\,8x^2+21x-44\,(\,4 & \\
\underline{8x^2+20x-48} & \\
x+4\,)\,2x^2+5x-12\,(\,x-3 & \\
\underline{2x^2+8x} & \\
-3x-12 & \\
\underline{-3x-12} &
\end{array}
$$

$\therefore$ G. C. M. $=2x(x+4)$.

Exercise 19.

Find the G. C. M. of

(1) $5x^2+4x-1$, $20x^2+21x-5$.
(2) $2x^3-4x^2-13x-7$, $6x^3-11x^2-37x-20$.
(3) $6a^4+25a^3-21a^2+4a$, $24a^4+112a^3-94a^2+18a$.
(4) $12(9x^3+9x^2-4x-4)$, $7(45x^3+54x^2-20x-24)$.
(5) $27x^6-3x^4+6x^3-3x^2$, $162x^6+48x^3-18x^2+6x$.
(6) $20x^3-60x^2+50x-20$, $32x^4-92x^3+68x^2-24x$.

39.—The algebraical proof of the above method, as applied to *numbers*, will be as follows:—

Let a and b be two numbers, of which a is the greater. The operation may be represented by—

$$\begin{array}{l} b\,)\,a\,(\,p \\ \quad \underline{pb} \\ \quad\ c\,)\,b\,(\,q \\ \qquad\ \underline{qc} \\ \qquad\ d\,)\,c\,(\,r \\ \qquad\qquad \underline{rd} \end{array}$$

p, q, r representing the several quotients,
c, d ,, the remainders,
and d being supposed to be contained exactly in c.

The numbers represented are all integral.

Then
$$c=rd$$
$$b=qc+d=qrd+d=(qr+1)d$$
$$a=pb+c=pqrd+pd+rd$$
$$=(pqr+p+r)d.$$

$\therefore$ a, b each contain d as a factor.

Also, d will be the *greatest* factor common to a, b.

For, suppose F to represent their greatest common factor; then since, from above, $c=a-pb$,
and a, b each contain F,
$\therefore$ c contains F.

Similarly, $d=b-qc$,
$\therefore$ d contains F.

d then contains the greatest factor common to a, b; but it was proved before to be itself contained in a, b;
$\therefore$ it must be that greatest factor.

The same proof will apply to the case of algebraical expressions already considered.

40.—The above method will be more clearly understood, it is distinctly borne in mind that every remainder in e course of the work contains as a factor of itself the . C. M. sought, and that this is the *highest* factor common that remainder and the preceding divisor.

The process may with advantage be represented as llows, F being the G. C. M. required:

$$\begin{array}{l}
nF\,)\,mF\,(\,p \\
\quad\;\; \underline{pnF} \\
\qquad\; lF\,)\,nF\,(\,q \\
\qquad\qquad \underline{qlF} \\
\qquad\qquad\;\; F\,)\,lF\,(\,l \\
\qquad\qquad\qquad\;\; \underline{lF}
\end{array}$$

41.—The method of finding the G. C. M. of two umbers has been seen in (37) to be exactly applicable lso to some algebraical expressions. But in many cases a odification of that method is needed. For example, in btaining the G. C. M. of

Ex. (3) $\qquad 3x^3+x^2-4x-20,\ x^3-x^2-4.$

$$\begin{array}{r}
x^3-x^2-4\,)\,3x^3+\ x^2-4x-20\,(\,3 \\
\underline{3x^3-3x^2\qquad\ -12} \\
4x^2-4x-\ 8
\end{array}$$

The first division ends here, because $4x^2$ has a lower index than x^3. But if as before $4x^2-4x-8$ is made the divisor, $4x^2$ is not contained in x^3 with an integral quotient. How is the difficulty to be overcome?

First, remembering that, as was shown above, the G. C. M. required is contained in the remainder at which we have arrived, $4x^2-4x-8$ or $4\,(x^2-x-2)$; and again that, since the given expressions contain no simple factor, there can be no simple factor in their G. C. M., it will be seen that if the simple factor 4 be removed, the G. C. M. must still be contained in x^2-x-2; and if this be taken for divisor, the process may be continued as before.

$$
\begin{array}{l}
x^2-x-2\,)\,x^3-x^2-4\,(\,x \\
\qquad\qquad\;\; \underline{x^3-x^2-2x} \\
\qquad\qquad\qquad 2\,)\,\underline{2x-4} \\
\qquad\qquad\qquad\quad x-2\,)\,x^2-x-2\,(\,x+1 \\
\qquad\qquad\qquad\qquad\qquad \underline{x^2-2x} \\
\qquad\qquad\qquad\qquad\qquad\quad x-2 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x-2}
\end{array}
$$

$\therefore$ G. C. M. $= x-2$.

A similar removal of the simple factor 2 is needed at the end of the second division.

Note.—It is clear that, since the simple factors here removed are not *common* to the given expressions, no notice need be taken of them in writing down the G. C. M.

Simple factors which are common to the given expressions must be separated as in (37), and their common factors must appear in the G. C. M.

Exercise 20.

Find the G. C. M. of

(1) $4x^2-8x-5$, $12x^2-4x-65$.

(2) $3a^3-5a^2x-2ax^2$, $9a^3-8a^2x-20ax^2$.

(3) $10x^3+x^2-9x+24$, $20x^4-17x^2+48x-3$.

(4) $8x^3-4x^2-32x-182$, $36x^3-84x^2-111x-126$.

(5) $5x^2(12x^3+4x^2+17x-3)$, $10x(24x^3-52x^2+14x-1)$.

(6) $9x^4y-x^2y^3-20xy^4$, $18x^3y-18x^2y^2-2xy^3-8y^4$.

42.—Another modification is sometimes required:

Ex. (4) Find the G. C. M. of $8x^2+2x-3$ and $6x^3+5x^2-2$.

The difficulty which presents itself here is of the same kind as before—viz. that neither first term is exactly contained in the other; but it is increased by there being in $8x^2+2x-3$ no simple factor which can be removed.

In this case, multiply the dividend by the simple factor 4, so as to make $8x^2$ exactly divisible into its first term.

$$6x^3+5x^2-2 \; . \; . \; . \; (A)$$

$$\begin{array}{lll}
 & & \underline{\quad 4 \quad} \\
(B). \; 8x^2+2x-3 &) & 24x^3+20x^2-8 \; (\; 3x \\
 & & \underline{24x^3+\;\;6x^2-9x} \\
(a) \; . \; . \; . \; . \; . \; . \; . \; . & & 14x^2+9x-8 \\
 & & \underline{\quad 4 \quad} \\
 & & 56x^2+36x-32 \; (\; 7 \\
 & & \underline{56x^2+14x-21} \\
 & & 11 \;) \; 22x-11 \\
 & & 2x-1 \;) \; 8x^2+2x-3 \; (\; 4x+3 \\
 & & \qquad \underline{8x^2-4x} \\
 & & \qquad\qquad 6x-3 \\
 & & \qquad\qquad \underline{6x-3}
\end{array}$$

$\therefore$ G. C. M. $= 2x-1$.

At (a) the same difficulty arises, to be obviated in the same way.

The method adopted here for facilitating the work is the reverse of that in the former case, a simple factor being *introduced* instead of removed. But the introduction of such a factor cannot affect the G. C. M. found, for the expression B contains no simple factor, and therefore has the same factors *in common with* A *as with* 4A, and therefore has with each the same *highest* common factor.

The same explanation will apply to (a).

Exercise 21.

Find the G. C. M. of

(1) $6x^2-x-15$, $9x^2-3x-20$.

(2) $12x^3-9x^2+5x+2$, $24x^2+10x+1$.

(3) $6x^3+15x^2-6x+9$, $9x^3+6x^2-51x+36$.

(4) $4x^3-x^2y-xy^2-5y^3$, $7x^3+4x^2y+4xy^2-3y^3$.

(5) $2a^3-2a^2-3a-2$, $3a^3-a^2-2a-16$.

(6) $12y^3+2y^2-94y-60$, $48y^3-24y^2-348y+30$.

(7) $9x(2x^4-6x^3-x^2+15x-10)$ and
$6x^2(4x^4+6x^3-4x^2-15x-15)$.

(8) $15x^4+2x^3-75x^2+5x+2$, $35x^4+x^3-175x^2+30x+1$.

(9) $21x^4-4x^3-15x^2-2x$, $21x^3-32x^2-54x-7$.

(10) $9x^4y-22x^2y^3-3xy^4+10y^5$, $9x^5y-6x^4y^2+x^3y^3-25xy^5$.

43.—The following will be the complete algebraical proof of the method for finding the G. C. M. of two expressions:—

Let M and N be two expressions, arranged by descending powers of one of their letters; let m and n represent the simple factors contained in them respectively, A and B the expressions when the simple factors have been separated; so that

$$M=mA,\ N=nB$$

(A and B containing then no simple factors whatever); and let f be the highest factor common to m and n.

Suppose the first term of A to be not of lower dimensions than that of B: then the operation may be represented by

$$\begin{array}{llllll}
B\,)\ \alpha A\ (\,p & & & & & \\
\quad\ \ \underline{pB} & & & & & \\
c\,)\ \overline{cC} & & & & & \\
\qquad \overline{C}\,)\ \beta B\ (\,q & & & & & \\
\qquad\qquad \underline{qC} & & & & & \\
\qquad\ d\,)\ \overline{dD} & & & & & \\
\qquad\qquad \overline{D}\,)\ C\ (\,r & & & & & \\
\qquad\qquad\quad\ \ \underline{rD} & & & & &
\end{array}$$

α, β being simple factors, introduced in order to make the first term of each dividend exactly divisible by the first term of the divisor;

c, d, simple factors of the remainders;

C, D, compound expressions containing no simple factors.

Suppose the division to end with the quotient r, so that

$$\begin{aligned} C&=rD \quad . \quad . \quad . \quad . \quad . \quad (1)\\ \text{then } \beta B&=qC+dD \quad . \quad . \quad . \quad . \quad (2)\\ \alpha A&=pB+cC \quad . \quad . \quad . \quad . \quad (3)\end{aligned}$$

By (1) the compound factor D is contained in C, and therefore in $qC+dD$ or βB, by (2).

But β is a simple quantity,

$\therefore D$ must be contained in B.

Similarly, by (3), D must be contained in A,

$\therefore D$ is a common factor of A and B.

Again, $\qquad cC=\alpha A-pB,$

$\therefore$ every factor common to A and B is contained also in cC, and therefore (since A and B contain only compound factors) in C.

And $$dD = \beta B - qC,$$
$\therefore$ every factor common to B and C is contained in dD, and therefore in D.

Therefore every factor common to A and B is contained in D.

Since, then, D contains every factor which is common to A and B, and is itself contained in both A and B, D must be the *highest* common factor of A and B.

Hence D is the highest compound factor common to M and N, and f is their highest simple factor;

$\therefore fD$ is the G. C. M. of M and N.

44.—It will be obvious that, if two expressions be each divided by their G. C. M., the quotients will be prime to one another.

45.—The G. C. M. of three expressions which are not resoluble on inspection, will be obtained by finding the G. C. M. of two of them, and then of that and the third expression:

For if A, B, C are three expressions,
and D the greatest common measure of A, B,
E " " " D, C,
D contains every factor which is contained in A, B; and E is the highest contained in D, C.

$\therefore E$ is the highest contained in A, B, C

Exercise 22.

Find the G. C. M. of

(1) $2x^2+3x-5$, $3x^2-x-2$, $2x^2+x-3$.
(2) x^3-1, x^3-x^2-x-2, $2x^3-x^2-x-3$.
(3) x^3-3x-2, $2x^3+3x^2-1$, x^3+1.
(4) $12(x^4-y^4)$, $10(x^6-y^6)$, $8(x^4y+xy^4)$.
(5) x^4+xy^3, x^3y+y^4, $x^4+x^2y^2+y^4$.
(6) $2(x^2y-xy^2)$, $3(x^3y-xy^3)$, $4(x^4y-xy^4)$, $5(x^5y-xy^5)$.

LEAST COMMON MULTIPLE OF COMPOUND EXPRESSIONS.

46.—The Least Common Multiple of two or more compound expressions is *the factor of lowest dimensions which exactly contains each.*

Two expressions which are prime to one another can be both contained only in their *product*, or in some *multiple of their product*: therefore their product must be their L. C. M.

When compound expressions can be separately resolved into their factors, their L. C. M. may be obtained by inspection, in the same manner as that of simple expressions. It will consist of all the factors which appear in either expression, each factor being written with its highest index.

Ex. (1) Find the L. C. M. of $2a^2+2ax$, $6(a^2-x^2)$, $3(a-x)^2$.

$$2a^2+2ax=2a(a+x),$$
$$6(a^2-x^2)=6(a+x)(a-x);$$

and there is also $3(a-x)^2$,

$$\therefore \text{ L. C. M.}=6a(a+x)(a-x)^2.$$

If one of the factors $(a-x)$ be multiplied into $(a+x)$, this may also be written as

$$6a(a^2-x^2)(a-x).$$

When the factors of the expressions are not readily obtainable, the expressions may be resolved by finding their G. C. M.

Ex. (2) Find the L. C. M. of $6x^3-11x^2y+2y^3$, and $9x^3-22xy^2-8y^3$.

$$\begin{array}{l}
\qquad\qquad\qquad\quad 9x^3-22xy^2-8y^3 \\
\qquad\qquad\qquad\quad 2 \\
6x^3-11x^2y+2y^3)\ 18x^3-44xy^2-16y^3\ (3 \\
\qquad\qquad\qquad\quad 18x^3-33x^2y+6y^3 \\
\qquad\qquad\qquad\quad 11y)\ 33x^2y-44xy^2-22y^3 \\
\qquad\qquad\qquad\qquad\quad 3x^2-4xy-2y^2)\ 6x^3-11x^2y+2y^3\ (2x-y \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 6x^3-8x^2y-4xy^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -3x^2y+4xy^2+2y^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -3x^2y+4xy^2+2y^3
\end{array}$$

Hence $6x^3-11x^2y+2y^3=(2x-y)(3x^2-4xy-2y^2)$,
and $9x^3-22xy^2-8y^3=(3x+4y)(3x^2-4xy-2y^2)$;

$\therefore$ L. C. M. $=(2x-y)(3x+4y)(3x^2-4xy-2y^2)$.

47.—To observe more clearly the relation between the G. C. M. and L. C. M. of two quantities, let A and B represent the quantities; let D be their G. C. M.; and let $A=P.D$, $B=Q.D$; P and Q being (44) prime to one another.

Then the L. C. M. of P and Q is $P.Q$, and therefore that of PD and QD is PQD.

Let M represent the L. C. M. of A and B;

then $M.D=PQD.D.=PD.QD=A.B$:

i.e. *the product of the* G. C. M. *and* L. C. M. *of two quantities* $=$ *product of the quantities themselves.*

Hence also $M=\dfrac{AB}{D}$ or $A.Q$ or $B.P$: i.e. the L. C. M. of two quantities can be found by *dividing the product of the quantities by their* G. C. M.;

Or, by *dividing one of the quantities by the* G. C. M., *and multiplying the result into the other quantity.*

Exercise 23.

Find the L. C M. of

(1) $2x(x+1)$, $3x^2(x+1)$.

(2) $6ac^2(a-c)$, $8a^2c(a+c)$.

(3) $12xy(x^2-y^2)$, $2x^2(x+y)^2$, $3y^2(x-y)^2$.

(4) $20(x^2-1)$, $24(x^2-x-2)$, $16(x^2+x-2)$.

(5) $4ab(a^2-3ab+2b^2)$, $5a^2(a^2+ab-6b^2)$.

(6) $(a-b)(b-c)$, $(b-c)(c-a)$, $(c-a)(a-b)$.

(7) $(a-b)(a-c)$, $(b-a)(b-c)$, $(c-a)(c-b)$.

(8) $a-b$, $a+b$, a^2-b^2, a^2+b^2.

(9) x^3-4x^2+3x, $x^4+x^3-12x^2$, $x^5+3x^4-4x^3$.

(10) x^2y-xy^2, $3x(x-y)^2$, $4y(x-y)^3$.

(11) $(a+b)^2-(c+d)^2$, $(a+c)^2-(b+d)^2$, $(a+d)^2-(b+c)^2$.

(12) $3x^3y-3xy^3$, $4x^4-4y^4$, $5x^5y-5xy^5$.

(13) $(2x-4)(3x-6)$, $(x-3)(4x-8)$, $(2x-6)(5x-10)$.

(14) $x-y$, $x+y$, x^2+y^2, x^4+y^4.

(15) $5(a^2-2ab)$, $10(ab+2b^2)$, $15(a^2b^2-4b^4)$.
(16) $5x^2+19x-4$, $10x^2+13x-3$.
(17) $12x^2+xy-6y^2$, $18x^2+18xy-20y^2$.
(18) x^4-2x^3+x, $2x^4-2x^3-2x-2$.
(19) $x^4+x^3+x^2+x$, $x^4-x^3+x^2-x$, x^4-x^2.
(20) $12x^2+2x-4$, $12x^2-30x-18$, $12x^2-28x-24$.

CHAPTER X.

FRACTIONS INVOLVING COMPOUND EXPRESSIONS.

3.—THE methods of dealing with compound expres-s given in the last three chapters may be applied to simplification of fractions, according to the ordinary hmetical rules.

Then fractions are to be multiplied together, it will be ıd best to resolve the numerators and denominators as as possible into their factors, so that common factors be readily observed.

ı addition and subtraction, the denominators should written in the factorial form.

x. (1) Reduce $\frac{6x^2-5x-6}{8x^2-2x-15}$ to its lowest terms.

The G. C. M. of the numerator and denominator will be found to be $2x-3$; and dividing both parts of the fraction by this, the result will be

$$\frac{3x+2}{4x+5}.$$

x. (2) Multiply $\frac{4(x^2-x-2)}{5x(x-3)}$ by $\frac{15x^2(x-1)}{2(x^2-1)}$.

$$\frac{4(x^2-x-2)}{5x(x-3)}=\frac{4(x+1)(x-2)}{5x(x-3)};$$

$$\frac{15x^2(x-1)}{2(x^2-1)}=\frac{15x^2(x-1)}{2(x+1)(x-1)}=\frac{15x^2}{2(x+1)};$$

and $\frac{4(x+1)(x-2)}{5x(x-3)}\times\frac{15x^2}{2(x+1)}=\frac{6x(x-2)}{x-3}$. *Ans.*

The factors 2, 5, x, $x+1$, which are common to a numerator and denominator, being struck out.

Ex. (3) Simplify $\frac{2a}{a^2-b^2}-\frac{3b}{a^2-ab}-\frac{a+3b}{ab+b^2}$.

The denominators are $(a+b)(a-b)$, $a(a-b)$, $(a+b)b$, whose L. C. M. is $ab(a+b)(a-b)$,

$$\therefore \frac{2a}{a^2-b^2}-\frac{3b}{a^2-ab}-\frac{a+3b}{ab+b^2}=$$

$$\frac{2a^2b-3b^2(a+b)-a(a+3b)(a-b)}{ab(a^2-b^2)},$$

$$=\frac{2a^2b-3ab^2-3b^3-a(a^2+2ab-3b^2)}{ab(a^2-b^2)},$$

$$=\frac{2a^2b-3ab^2-3b^3-a^3-2a^2b+3ab^2}{ab(a^2-b^2)},$$

$$=-\frac{a^3+3b^3}{ab(a^2-b^2)}. \quad \textit{Ans.}$$

It is sometimes best in addition, to add two or more of the fractions together first.

Ex. (4) $\frac{2a+b}{a-b}-\frac{2a-b}{a+b}-\frac{6ab}{a^2+b^2}$,

$$=\frac{(2a+b)(a+b)-(2a-b)(a-b)}{a^2-b^2}-\frac{6ab}{a^2+b^2},$$

$$=\frac{2a^2+3ab+b^2-(2a^2-3ab+b^2)}{a^2-b^2}-\frac{6ab}{a^2+b^2},$$

$$=\frac{2a^2+3ab+b^2-2a^2+3ab-b^2}{a^2-b^2}-\frac{6ab}{a^2+b^2},$$

$$=\frac{6ab}{a^2-b^2}-\frac{6ab}{a^2+b^2},$$

$$=\frac{6ab(a^2+b^2)-6ab(a^2-b^2)}{a^4-b^4},$$

$$=\frac{12ab^3}{a^4-b^4}. \quad \textit{Ans.}$$

Exercise 24.

Express in the form of mixed numbers

(1) $\frac{2x^2-5x-2}{x-4}$. (2) $\frac{a^2+b^2}{a-b}$. (3) $\frac{5x^3-x^2+5}{5x^2+4x-1}$.

Express as simple fractions—

(4) $a-1+\frac{1}{a+1}$. (5) $x+5-\frac{2x-15}{x-3}$. (6) $2a-b-\frac{2ab}{a+b}$.

Simplify—

(7) $\frac{x^2-1}{4x(x+1)}$. (8) $\frac{(a+b)^2}{a^2-ab-2b^2}$. (9) $\frac{3ab(a^2-b^2)}{4(a^2b-ab^2)^2}$.

(10) $\frac{a}{a+b}+\frac{a}{a-b}$. (11) $\frac{a}{(a+b)b}-\frac{b}{(a-b)a}$.

(12) $\frac{2a(a-x)}{3(a^2+ax)}\times\frac{9(a^2-x^2)}{4(ax-x^2)}$. (13) $\frac{6x^3-11x^2y+3xy^2}{6x^2y-5xy^2-6y^3}$.

(14) $\frac{a^2-3a-4}{a^2-1}\div\frac{a^2-16}{a^2-a}$. (15) $\frac{(a+b)^2-c^2}{a^2+ab-ac}$.

(16) $\frac{2x-3y}{x^2y(x+y)}+\frac{3x-2y}{xy^2(x+y)}$.

(17) $\frac{3(a+2b)^2(a-3b)b}{a(2a+4b)(a^2-9b^2)}$. (18) $\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}$.

(19) $\frac{2}{a}-\frac{3}{b}+\frac{4}{a+2b}$. (20) $\frac{5}{2x(x-1)}-\frac{3}{4x(x-2)}$.

(21) $\frac{a}{a-x}-\frac{x}{a+2x}-\frac{a^2+x^2}{(a-x)(a+2x)}$.

(22) $\frac{3}{(a-b)(b-c)}-\frac{4}{(a-b)(a-c)}+\frac{6}{(a-c)(b-c)}$.

(23) $\frac{3}{(x-1)(x-2)}-\frac{2}{(x-1)(x-3)}-\frac{1}{(x-2)(x-3)}$.

(24) $\frac{x-2y}{x(x-y)}-\frac{2x+y}{y(x+y)}-\frac{2x}{x^2-y^2}$.

(25) $\frac{a-b}{(a+b)x}-\frac{a-b}{(a+b)y}-\frac{(a-b)(x+y)}{(a+b)xy}$.

(26) $\frac{3x}{(x+y)^2}-\frac{x+2y}{x^2-y^2}+\frac{3y}{(x-y)^2}$.

(27) $\left(\frac{a}{x}-\frac{x}{a}\right)\left(\frac{a}{x}+\frac{x}{a}\right)\left(1-\frac{x-a}{x+a}\right)$.

(28) $\left(\frac{a}{x}+\frac{x}{a}-2\right)\left(\frac{a}{x}+\frac{x}{a}+2\right)\div\left(\frac{a}{x}-\frac{x}{a}\right)^2$.

(29) $\frac{2x}{x^2-x+1}-\frac{3x}{x^2+x+1}+\frac{x^2(x-5)}{x^4+x^2+1}$.

(30) $\frac{a-c}{(a+b)^2-c^2}-\frac{a-b}{(a+c)^2-b^2}$.

(31) $\dfrac{7}{3x^2-9x-12}+\dfrac{5}{6x^2-6}+\dfrac{1}{2x^2-8x}$.

(32) $\dfrac{a}{a+x}-\dfrac{x}{a-x}+\dfrac{2ax}{a^2+x^2}$.

(33) $\dfrac{b}{(a-b)(b-c)}+\dfrac{c}{(b-c)(c-a)}+\dfrac{a}{(c-a)(a-b)}$.

(34) $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}$.

(35) $\left(\dfrac{1}{x-y}-\dfrac{x}{x^2-y^2}\right)\div\left(\dfrac{x}{xy+y^2}-\dfrac{y}{x^2+xy}\right)$.

(36) $\dfrac{a^2-(b+c+d)^2}{(a-b)^2-(c+d)^2}$.

(37) $\dfrac{a+b}{ax+by}-\dfrac{a-b}{ax-by}+\dfrac{ab(x-y)}{a^2x^2-b^2y^2}$.

(38) $\dfrac{1}{(a-b)^2-c^2}+\dfrac{1}{(a-c)^2-b^2}+\dfrac{1}{(b-c)^2-a^2}$.

(39) $\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{2x^3y}{x^4-y^4}-\dfrac{2xy^7}{x^8-y^8}$.

(40) $\dfrac{x-4+\dfrac{6}{x+1}}{x-\dfrac{6}{x-1}}\times\dfrac{1-\dfrac{x+5}{x^2-1}}{(x-1)(x-2)}$.

(41) $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$. (42) $\dfrac{(x^2-y^2)\times(2x^2-2xy)}{4(x-y)^2\div\dfrac{xy}{x+y}}$.

(43) $\left(\dfrac{ab}{x^2+(a+b)x+ab}-\dfrac{ac}{x^2+(a+c)x+ac}\right)$
$$\div\dfrac{b-c}{x^2+(b+c)x+bc}.$$

(44) $\dfrac{y}{(x-y)(x-z)}+\dfrac{x}{(y-x)(y-z)}+\dfrac{x+y}{(z-x)(z-y)}$.

(45) $\left(1+\dfrac{x^2+y^2-z^2}{2xy}\right)\div\left(1-\dfrac{x^2-y^2-z^2}{2yz}\right)$.

(46) $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-x)(y-z)}-\dfrac{1}{xyz}$.

(47) $\dfrac{a+b}{a^2+ab+b^2}+\dfrac{a-b}{a^2-ab+b^2}-\dfrac{2(a^2b-ab^2)}{a^4+a^2b^2+b^4}$.

(48) $$\frac{x^2+\left(\frac{a}{b}+\frac{b}{a}\right)xy+y^2}{x^2+\left(\frac{a}{b}-\frac{b}{a}\right)xy-y^2}.$$

(49) If $\frac{x}{y+z}=a, \frac{y}{z+x}=b, \frac{z}{x+y}=c$, show that $\frac{x^2}{a(1-bc)}$ $=\frac{y^2}{b(1-ac)}=\frac{z^2}{c(1-ab)}$, by expressing each fraction in terms of x, y, z.

(50) Find the value of $\frac{x+y-1}{x-y+1}$, when $x=\frac{a+1}{ab+1}$,

and $y=\frac{ab+a}{ab+1}$.

CHAPTER XI.

SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

49.—SIMPLE EQUATIONS with one unknown quantity involving compound expressions are to be worked on the same principle as the easier Equations given in Chapter IV.

Ex. (1) Solve $\dfrac{2x-3}{3x-2}=\dfrac{2(x-1)}{3x+2}$

Clearing the equation of fractions, we have

$$(2x-3)(3x+2)=2(x-1)(3x-2),$$

or, $$6x^2-5x-6=6x^2-10x+4;$$

Transposing $$6x^2-6x^2-5x+10x=6+4,$$

i.e. $$5x=10,$$

$$\therefore x=2.$$

When the denominators of the fractions involved contain both simple and compound factors, it is frequently best to multiply the equation out by the simple factors first, and then to collect the integral terms before multiplying by the compound factors.

Ex. (2) Solve $\dfrac{6x+7}{9}+\dfrac{7x-13}{3(2x+1)}=\dfrac{2(x+2)}{3}.$

Multiplying by 9, the L. C. M. of the simple factors in the denominators,

$$6x+7+\frac{21x-39}{2x+1}=6x+12;$$

Transposing. $$\frac{21x-39}{2x+1}=5;$$

Multiplying now by $2x+1$,

$$21x-39=10x+5;$$

Transposing, $$11x=44,$$

$$\therefore \quad x=4.$$

Brackets must be removed in the course of simplification.

Ex. (3) Solve $\frac{2}{5}(x-4)-\frac{1}{2}\{x-\frac{1}{3}(2x-1)\}=\frac{1}{2}(x-3)$.

Multiply by 10 the L. C. M. of the denominators outside the brackets;

$$\text{then } 4(x-4)-5\{x-\tfrac{1}{3}(2x-1)\}=5(x-3);$$

$$\text{or, } 4x-16-5x+\tfrac{5}{3}(2x-1)=5x-15;$$

$$\text{Transposing, } \quad \tfrac{5}{3}(2x-1)=6x+1;$$

$$\text{Multiply by } 3, \quad 5(2x-1)=18x+3,$$

$$\text{or, } \quad 10x-5=18x+3,$$

$$-8x=8,$$

$$\therefore x=-1.$$

Complex fractions should be separately simplified.

Ex. (4) Solve $\dfrac{\frac{4}{3}(3x-4)-\frac{3}{4}(4x-3)}{\frac{5}{6}(x-1)}=\dfrac{(x+1)}{5x}+1$.

The fraction on the left side will be best simplified by multiplying its numerator and denominator by 12.

$$\text{Then, } \frac{16(3x-4)-9(4x-3)}{10(x-1)}=\frac{x+1}{5x}+1;$$

$$\text{or, } \frac{48x-64-36x+27}{10(x-1)}=\frac{x-1}{5x}+1;$$

$$\text{or, } \frac{12x-37}{10(x-1)}=\frac{x+1}{5x}+1;$$

$$\therefore \frac{12x^2-37x}{x-1}=2x+2+10x$$

$$=12x+2,$$

$$12x^2-37x=12x^2-10x-2,$$

$$-27x=-2$$

$$x=\frac{2}{27}.$$

Note.—**The simplification of the complex fraction changes only its *form*, not its *value*, so that it may be performed without altering the rest of the equation.**

Equations often involve *literal* coefficients, and letters (*a*, *b*, *c*, &c.) which are considered as known quantities.

Ex. (5) Solve—

$$(x-a)(x-b)-(x-b)(x-c)=2(x-a)(a-c).$$

$$(x^2-ax-bx+ab)-(x^2-bx-cx+bc)=2(ax-cx-a^2+ac),$$

or $x^2-ax-bx+ab-x^2+bx+cx-bc=2ax-2cx-2a^2+2ac.$

Transposing $-3ax+3cx=-2a^2+2ac-ab+bc,$

or, $-3(a-c)x=-(2a+b)(a-c),$

$$x=\frac{2a+b}{3}.$$

The solution of an equation will often be facilitated by particular arrangements and combinations of the terms, and advantage should be taken of any such means of simplifying the work.

For instance, the equation

Ex. (6) $\dfrac{x-1}{x-3}+\dfrac{x-6}{x-8}=\dfrac{x-2}{x-4}+\dfrac{x-5}{x-7}$

might be solved by multiplying by the four denominators; but the work will be simplified by transposing two of the fractions,

$$\frac{x-1}{x-3}-\frac{x-2}{x-4}=\frac{x-5}{x-7}-\frac{x-6}{x-8},$$

and expressing each side as a single fraction,

$$\frac{-2}{(x-3)(x-4)}=\frac{-2}{(x-7)(x-8)}.$$

$$\therefore (x-3)(x-4)=(x-7)(x-8),$$

$$x^2-7x+12=x^2-15x+56,$$

$$8x=44,$$

$$x=5\frac{1}{2}.$$

The work might have been still further simplified by writing the equation at first as—

$$1+\frac{2}{x-3}+1+\frac{2}{x-8}=1+\frac{2}{x-4}+1+\frac{2}{x-7},$$

which gives $\dfrac{1}{x-3}+\dfrac{1}{x-8}=\dfrac{1}{x-4}+\dfrac{1}{x-7},$

or, $\dfrac{1}{x-3}-\dfrac{1}{x-4}=\dfrac{1}{x-7}-\dfrac{1}{x-8},$ &c.

Exercise 25.

Solve the equations—

(1) $\frac{2(x-3)}{x+4}=\frac{1}{4}$. (2) $(x-4)(x+6)=(x+3)(x-2)$.

(3) $\frac{x}{x+1}+\frac{2x}{x-1}=3$. (4) $\frac{x-4}{2x+1}=\frac{2x-7}{4x-5}$.

(5) $(2x-3)(4x-5)=8(x-1)(x-2)$.

(6) $\frac{1-x}{2}\cdot\frac{2-x}{3}=\frac{5}{9}\cdot\frac{3-2x}{4}\cdot\frac{4-3x}{5}-1$.

(7) $\frac{1}{7}(4x-5)-\frac{1}{3}(2x+7)=\frac{1}{5}(x-30)$.

(8) $\frac{1}{3}(x-1)-\frac{1}{4}\{2x-(x-3)\}=2$.

(9) $\frac{2}{3}\{2x-\frac{1}{4}(x-5)\}=\frac{5}{12}\{x+\frac{1}{2}(x-8)\}$.

(10) $\frac{3-\frac{4x}{9}}{4}=\frac{1}{4}-\frac{\frac{7x}{9}-3}{10}$.

(11) $\frac{9x+20}{36}=\frac{4(x-3)}{5x-4}+\frac{x}{4}$.

(12) $\frac{9(2x-3)}{14}+\frac{11x-1}{3x+1}=\frac{9x+11}{7}$.

(13) $\frac{7}{x-1}=\frac{6x+1}{x+1}-\frac{3(1+2x^2)}{x^2-1}$.

(14) $\frac{1}{2}(x+9)+\frac{1}{4}(5x+13)=9-\frac{3}{8}(3x+11)$.

(15) $\frac{2}{7}(x-3)-\frac{6}{5}(x-2)=\frac{7}{10}(3x-1)$.

(16) $x+\frac{x}{x-1}=\frac{(x-2)(x+4)}{x+1}$.

(17) $\frac{\frac{1}{2}(x-1)+\frac{1}{3}(2x+7)}{3(x+2)}=\frac{3}{x+2}+\frac{1}{27}$.

(18) $(x+2)(x-6)-(x-2)^2=3x-1$.

(19) $(x+1\cdot4)(x+\cdot6)=(x-2\cdot5)(x+\cdot5)$.

(20) $\frac{x-3}{4(x-1)}=\frac{x-5}{6(x-1)}+\frac{1}{9}$.

(21) $\frac{x-1}{x-2}+\frac{x-5}{x-6}=\frac{x-2}{x-3}+\frac{x-4}{x-5}.$

(22) $\frac{1}{x-1}+\frac{2}{2x-1}=\frac{6}{3x-1}.$ (23) $\frac{x}{a}+\frac{x}{b}=c.$

(24) $\frac{x-a^2}{b}+\frac{x-b^2}{a}=a+b.$ (25) $\frac{1+x}{1-x}=1+\frac{1}{a}.$

(26) $\frac{1}{2(x-3)}-\frac{1}{3(x-2)}=\frac{x-1}{(x-2)(x-3)}.$

(27) $\frac{\frac{1}{4}(x-1)+\frac{1}{10}(x+5)}{\frac{1}{5}(2x-5)}=\frac{\frac{5}{6}(x-4)}{\frac{1}{4}(2x+1)-\frac{1}{3}(x+2)}.$

(28) $\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+5}=\frac{9}{3x+8}.$

(29) $\frac{1}{x}+\frac{2}{x-1}=\frac{3}{x-2}-\frac{8}{x(2x-7)}.$

(30) $\frac{1}{a}\{x-\frac{1}{b}(x-c)\}=\frac{1}{b}\{x-\frac{1}{a}(x+c)\}.$

(31) $\frac{a}{bx}+\frac{b}{cx}+\frac{c}{dx}=d.$

(32) $1-\frac{2(2x+3)}{9(7-x)}=\frac{6}{7-x}-\frac{5x+1}{4(7-x)}.$

(33) $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}.$

(34) $\frac{3x+7}{21}-\frac{x+\frac{9}{10}}{3}=\frac{(x+2)^2-\frac{7}{2}}{7(x+1)}-\frac{x+\frac{4}{5}}{3}-\frac{1}{105}.$

(35) $\frac{3}{4}+\frac{12-11x}{8x+20}-\frac{5x+2}{12x+30}=\frac{8-x}{6x+15}-\frac{1}{3}.$

(36) $\frac{1}{5}-\frac{3}{x-1}=\frac{2+\frac{x+4}{1-x}}{3}.$

(37) $(x-a)^3=2x^3-ax^2-x(x+a)^2.$

(38) $\frac{17}{x+3}-4=5\left\{\frac{21+2x}{3x+9}-2\right\}.$

(39) $(x+\cdot 1)(x+\cdot 2)(x+\cdot 3)=(x+\cdot 4)(x+\cdot 5)(x-\cdot 3).$

(40) $\frac{1}{x+1}+\frac{1}{x-1}+\frac{1}{x+3}+\frac{1}{x-3}=\frac{4x}{x^2-5}.$

(41) $\dfrac{x-\frac{3}{2}}{\frac{3}{2}(x-1)}+\dfrac{x-\frac{5}{2}}{\frac{5}{2}(x+1)}=1+\dfrac{1}{15\left(1-\frac{1}{x^2}\right)}.$

(42) $\dfrac{a}{bx}+\dfrac{c}{dx}+\dfrac{e}{fx}-g=h.$

(43) $\dfrac{x}{x-a}+\dfrac{x}{x-b}=\dfrac{2(x^2-ab)}{(x-a)(x-b)}.$

(44) $\dfrac{a}{b+cx}-\dfrac{b}{a+cx}=\dfrac{a-b}{c+cx}.$

(45) $(x-a)^3-(x-b)^3+3(a-b)(x^2+ab)=0.$

(46) $\dfrac{x+4}{2x+4}+\dfrac{1}{x+4}+\dfrac{x}{2x-4}=\dfrac{x(x+6)}{x^2+2x-4}-\dfrac{1}{x}.$

(47) $(a+b)(a+c)(x+a)-(a+b)(b+c)(x+b)=(a-b)(b-c)(c-a).$

(48) $\dfrac{a+b}{a-b}x-\dfrac{a+c}{a-c}x+\dfrac{b-c}{b+c}(x-a)=0.$

(49) $\dfrac{a}{(x-a)(x-c)}-\dfrac{c}{(a-c)(a-x)}+\dfrac{x}{(c-x)(c-a)}=\dfrac{1}{a-c}.$

(50) $\dfrac{1}{x+a}+\dfrac{1}{x+b}+\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{4x}{x^2-\frac{1}{2}(a^2+b^2)}.$

CHAPTER XII.

PROBLEMS.

50.—The following problems differ from those of Chapter VI. only in being rather more difficult, and often involving compound expressions:—

Ex. (1) A sum of $4\frac{1}{2}$ guineas is made up of shillings, florins, and halfcrowns: the number of shillings and florins together being 30, and that of florins and halfcrowns together 36, how many coins of each kind are there?

Let x be the number of florins;
then $30-x=$,, shillings,
and $36-x=$,, halfcrowns.

The value of the florins $=4x$ sixpences.
,, ,, shillings $=2(30-x)$,,
,, ,, halfcrowns $=5(36-x)$,,
,, ,, whole sum $=189$,,

$$\therefore\ 4x+2(30-x)+5(36-x)=189,$$
$$4x+60-2x+180-5x=189,$$
$$3x=51.$$
$x=17=$ number of florins,
$30-x=13=$,, shillings,
$36-x=19=$,, halfcrowns.

Ex. (2) A number consists of three digits, those in the tens' and hundreds' places being respectively 2 and 3 less than that in the unit's place. If the order of the digits were inverted, and 62 subtracted, the result would be double of the original number: find it.

Let x be the digit in the unit's place.
then $x-2=$,, tens' ,,
and $x-3=$,, hundreds' ,,
Therefore the number is $100(x-3)+10(x-2)+x$
$$=111x-320.$$

If the digits were inverted, the number would become

$$100x+10(x-2)+(x-3)=111x-23.$$

$$\therefore\ 111x-23-62=2(111x-320),$$
$$111x-23-62=222x-640,$$
$$111x=555,$$
$$x=\ 5,$$

$\therefore$ the number is 235.

Ex. (3) Find three numbers in the proportion of 2, 3, 6, such that when each is diminished by 4, the products of the first and second, and of the first and third, may be together 32 less than that of the second and third.

Let $2x$, $3x$, $6x$ be the numbers, then

$$(2x-4)(3x-4)+(2x-4)(6x-4)=(3x-4)(6x-4)-32,$$
$$6x^2-20x+16+12x^2-32x+16=18x^2-36x+16-32,$$
$$16x=48,$$
$$x=\ 3,$$

$\therefore$ the numbers are 6, 9, 18.

Exercise 26.

(1) A sum of 36*s.* is divided among 17 persons, some of whom receive two shillings, and the rest half-a-crown; how many receive each sum?

(2) A has 6*s.* more than B, and if he receive 10*s.* from B, will then have twice as much as B: how much has each?

(3) A father's age exceeds the son's by 31 years, and is as much below 60 as the son's is above 19; what is the age of each?

(4) Find four successive numbers, such that half of the first, a third of the second, a fourth of the third, and a fifth of the fourth, may together amount to 81.

(5) In going a certain distance, a train travelling 35 miles an hour takes two hours less than one travelling 25 miles an hour; determine the distance.

(6) A rectangle whose length is 5 feet more than its breadth, would have its area increased by 22 feet if its

length and breadth were each made a foot more; find its dimensions.

(7) A and B have 24s. between them, A and C have 30s., and B and C have 32s.; how much has each?

(8) A draper sold two pieces of cloth, by one of which he gained £3 less than by the other, and his whole gain was £2 10s. more than treble the less gain; what did he gain on each piece?

(9) Divide 35 into two parts such that a fourth of the first may exceed half the second by 5.

(10) A rectangle has its length and breadth respectively 5 feet longer and 3 feet shorter than the side of the equivalent square; find its area.

(11) The members of a club subscribe as many shillings each as there are members: if there had been 12 more members, the subscription from each would have been half a sovereign less in order to amount to the same sum; how many members are there?

(12) Two persons set out from the same place in opposite directions, the rate of one of them per hour being a mile less than double the rate of the other, and in 4 hours they were 32 miles apart; determine their rates.

(13) Divide a sovereign among four persons, so that if the shares were increased by 1s., 2s., 3s., and 4s. respectively, they would become equal.

(14) Divide a sovereign between two persons, so that if the shares were increased by a shilling and half-a-crown respectively, the latter would be double the former.

(15) In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 less, than the number sunk: fifteen escaped, and the fleet consisted of eight times the number sunk; what was the strength of the fleet?

(16) Find a number the double of whose defect from 50 exceeds by 5 the treble of its excess above 30.

(17) From two casks of equal size quantities are drawn in the proportion of 5 to 6: it is then found that the whole quantity drawn is 3 gallons less than either cask would hold, and that the quantity left in one cask is 3 gallons less than in the other; find the size of each cask.

(18) The number of shillings in a bag is the square of the number of sovereigns: had the sovereigns been guineas, but one less in number, and the number of shillings the square of this, the amount would have been 30*s.* less; what is the amount?

(19) A and B travelling, each with £80, meet with robbers, who take from A twice as much as from B and £5 over, and leave A £13 less than half of what they leave B; how much do they take from each?

(20) The length of a rectangle is an inch less than double its breadth, and when a strip 3 inches wide is cut off all round, the area is diminished by 210 inches; determine the size of the rectangle at first.

(21) When a sum of money was divided among A, B, and C, A received 30*s.* less than the half, B a sovereign more than the third, and C a guinea less than the fourth; what was the sum?

(22) How many lbs. at 5*s.* 3*d.* must be added to 24 lbs. at 4*s.* 1*d.* to make the whole worth 4*s.* 7*d.* per lb.?

(23) A cylinder of wood encloses a cylinder of iron, the whole containing 30 cubic inches, and weighing 52 oz.: if the wood weighs ·5 oz. per cubic inch, and the iron 4·2 oz., how many inches of iron are there?

(24) A workman was engaged for 50 days at the rate of 3*s.* 6*d.* per day, with the condition that for every day he should be idle, instead of receiving he was to pay 1*s.*: on the whole he received £7 17*s.*; on how many days was he idle?

(25) A number consists of two digits, of which that in the tens' place is 3 more than the other, and the number itself is 7 times the sum of the digits; find the number.

(26) Divide 100 into four parts, such that the first increased by 4, the second diminished by 4, the third multiplied by 4, and the fourth divided by 4, may be all equal.

(27) A can do half as much work as B, and B half as much as C; how long would each take to do separately some work which together they can do in 10 hours?

(28) Of three workmen (A, B, C), B would take 2 hours more than A to complete some work, and C, who can do as much as A and B together, would take 20 minutes more than half of A's time; how long would each take?

(29) From each of two towns 45 miles apart, a traveller sets out at 9 o'clock towards the other town, and the rate of one traveller per hour is 5 miles more than three-fourths of that of the other: they meet at a quarter before 1 o'clock—at what distance from each town?

(30) In half an hour's time the number of minutes to 12 will be five times as many as the number of minutes past 12 half an hour ago: what is the time?

(31) What number must be added to the numerator and denominator of $\frac{29}{70}$ to make it equal to $\frac{3}{4}$?

(32) A mass of tin and lead weighing 180 lbs. loses 21 lbs. when weighed in water, and it is known that 37 lbs. of tin lose 5 lbs., and 23 lbs. of lead lose 2 lbs. in water; what are the weights of tin and lead in the mass?

(33) Divide a line a into two parts, such that the difference of their squares may be equal to b^2.

(34) Some land was sold at a loss for £a: if it had been sold for £b, the gain would have been n times the loss; what did it cost?

(35) One hand of a watch makes a revolution in a hours, and the other in b hours: if they start together, when will they be together again? $(a > b)$.

(36) Find three successive numbers, such that a third of the greatest may be 2 less than a fifth of the other two together.

(37) A train leaves P at 11 A.M. for Q, and travels at the rate of 25 miles an hour: another train leaves R at noon, and runs through P to Q at the rate of 35 miles an hour, arriving at Q 24 minutes later than the first: the distance from R to P being 21 miles, find the distance from R to Q.

(38) Of two guns (A and B), A fires two rounds more than B in half an hour; and if A then cease firing for 10 minutes, the number of rounds fired by it is four-fifths of that fired by B; what number of rounds is fired by each?

(39) A person bought for £1350 a piece of land, from which he cut off four-ninths for himself. At the cost of £50 he made a road which took one-tenth of the remainder, and then sold the rest at 6d. per square yard more than

ouble of what it cost him, thereby clearing his outlay and 100 besides; how much land did he buy, and what was the cost price per yard?

(40) A and B ran a race. After running four minutes the same uniform pace at which each started, the distance between them was $\frac{1}{440}$ of the whole course. They continued to run at the same rate for one minute more, and then B, who was last, quickened his speed 5 yards a minute, and was half a yard beyond the winning post when A reached it—A having run the whole course in six minutes at an uniform rate. What was the length of the course?

CHAPTER XIII.

SIMULTANEOUS SIMPLE EQUATIONS.

51.—The equations whose solutions have been obtained in the preceding chapters have involved only *one* unknown quantity, and in every case only one solution has been obtained. But an equation may be proposed for solution containing *two or more* unknown quantities; as

$$2x-3y=4 \quad . \quad . \quad . \quad . \quad (1);$$

and in this case it will be seen that, unless some other condition be assigned, the number of solutions will be unlimited. For any values (1, 2, 3, &c.) may be given to y, and then corresponding values ($3\frac{1}{2}$, 5, $6\frac{1}{2}$, &c.) will be found for x; and any pair of these values ($3\frac{1}{2}$, 1; 5, 2; &c.) substituted for x and y will satisfy the equation.

But a second independent simple equation might be also given, and values of x and y required which would satisfy *both*; and in such a case it will always be found that one pair of roots, and only one, may be obtained.

Equations which are to be satisfied by the same values of the unknown quantities are said to be *simultaneous*.

An equation simultaneous with the above may, for example, be

$$3x+2y=32 \quad . \quad . \quad . \quad . \quad (2)$$

52.—In investigating a method for solving simultaneous equations, it must be borne in mind that the values of x and y, though as yet unknown, are determinate, and are the same for both equations. The equations therefore may be combined in various ways, and the resulting equations, if obtained by treating the two sides of these alike, will still be satisfied by those same values.

(1) and (2) may, for instance, be added together, and it will be clear that the result

$$5x-y=36,$$

since it consists of equal quantities added to equals, must be true of the values of x and y, which satisfy (1) and (2).

So (1) and (2) might be subtracted one from the other, or multiplied or divided one by the other, and the result would in each case be satisfied by the same values of x and y as before.

Of these combinations, there are some which will give as result an equation containing only one of the unknowns; and such having been found, that unknown may be determined; and then by substituting in one of the given equations, the other unknown may be also determined.

The process of obtaining from the two given equations an equation from which one of the unknowns has disappeared is known as *elimination*, that letter being said to be *eliminated*.

The elimination may be best performed by three methods, and the preference to be given to either depends upon the form of the given equations. For a pair of simple equations the first is generally the best.

Each method may be applied to the above pair of equations:

$$\begin{cases} 2x-3y=4 \quad . \quad . \quad . \quad . \quad (1) \\ 3x+2y=32 \quad . \quad . \quad . \quad . \quad (2) \end{cases}$$

First Method.—Multiply each equation by such a number as to make the coefficients of one of the unknowns the same in both.

In this case, to make the coefficient of y the same in both, multiply (1) by 2, and (2) by 3:

$$4x-6y=8 \quad . \quad . \quad . \quad . \quad (3)$$
$$9x+6y=96 \quad . \quad . \quad . \quad . \quad (4)$$

Add (3) and (4):

$$13x=104,$$
$$\therefore x=8.$$

Write now this value for x in (1):

$$16-3y=4,$$
$$\therefore y=4.$$

Or;

To eliminate x, multiply (1) by 3, and (2) by 2:

$$6x-9y=12 \quad . \quad . \quad . \quad . \quad (5)$$
$$6x+4y=64 \quad . \quad . \quad . \quad . \quad (6)$$

Subtract (5) from (6):

$$13y=52,$$
$$\therefore y=4.$$

Substitute in (1):

$$2x-12=4,$$
$$\therefore x=8.$$

Note.—(3) and (4) are *added* because in them $6y$ has opposite signs: (5) and (6) are *subtracted* because in them $6x$ has the same sign—the object in each case being merely to get rid of the letter.

Second Method.—Transpose each equation, so as to have one of the unknowns alone on one side:

$$2x=3y+4,$$
$$3x=-2y+32,$$

and divide one of these by the other,

$$\frac{2}{3}=\frac{3y+4}{-2y+32},$$
$$\therefore -4y+64=9y+12,$$
$$-13y=-52,$$
$$y=4.$$

Third Method.—Obtain from one of the equations one of the unknowns in terms of the other, and substitute this value for it in the other equation.

From (1) $\quad x=\frac{3y+4}{2},$

Substituting for x in (2),

$$\frac{3(3y+4)}{2}+2y=32,$$
$$9y+12+4y=64,$$
$$13y=52,$$
$$y=4.$$

Ex (2) Solve $7x+3y=11$, $4x+12y=-4$.

Here the coefficient 12 is 4 times the 3,

$\therefore$ multiplying (1) by 4

$$28x+12y=44,$$

and subtracting $4x+12y=-4$,

we have $24x \quad =48$,

$$\therefore x=2,$$

$\therefore$ from (1) $14+3y=11$,

$$y=-1.$$

Each equation must be simplified, if necessary, before the elimination is performed.

Ex. (3) Solve $(x-1)(y+2)=(x-3)(y-1)+8$ . . (1)

$$\frac{2x-1}{5}-\frac{3(y-2)}{4}=1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(1) becomes $xy+2x-y-2=xy-x-3y+3+8$,

$$2x+x-y+3y=3+8+2,$$

$$3x+2y=13 \quad . \quad . \quad . \quad . \quad (3)$$

(2) gives $8x-4-15y+30=20$,

$$8x-15y=-6 \quad . \quad . \quad . \quad . \quad (4)$$

(3) $\times$ 8 gives $24x+16y=104$.

(4) $\times$ 3 $24x-45y=-18$,

subtracting $61y=122$,

$\therefore y=2$.

Substitute in (3) $3x+4=13$,

$\therefore x=3$.

Exercise 27.

Solve—

(1) $2x+3y=7$; $4x-5y=3$.

(2) $x-2y=4$; $2x-y=5$.

(3) $7x+2y=30$; $y=3x+2$.

(4) $3x=4y-5$; $4x=5y+1$.

(5) $6x+5y=4$; $5x+6y=1\frac{1}{2}$.

(6) $\frac{x}{2}+\frac{y}{3}=4$; $\frac{5x}{6}-\frac{2y}{3}=3$.

(7) $\frac{x-4}{5}-\frac{y+2}{10}=0$; $\frac{x}{6}+\frac{y-2}{4}=3$.

(8) $(x+1)(y+2)-(x+2)(y+1)=-1$;
$3(x+3)-4(y+4)=-8$.

(9) $\frac{2}{x+3}=\frac{3}{y-2}$; $5(x+3)=3(y-2)+2$.

(10) $\cdot 2x-\cdot 7y=3$; $\cdot 1x-\cdot 2-\cdot 3y=0$.

(11) $\frac{y-1}{4x}-2=\frac{2y+5}{3x}+2$; $\frac{x-3}{4y}-\frac{1}{10}=\frac{2(2x-1)}{3y}+\frac{1}{10}$.

(12) $ax+by=c$; $ax+cy=d$.

(13) $\frac{x}{a}+\frac{y}{b}=c$; $\frac{x}{b}+\frac{y}{a}=-c$.

(14) $\frac{x-a}{y-b}=c$; $a(x-a)+b(y-b)+abc=0$.

(15) $\frac{1}{x}+\frac{2}{y}=4,\ \frac{3}{x}-\frac{2}{y}=4.$

(16) $\frac{2}{x-1}+\frac{3}{y+1}=\frac{4}{(x-1)(y+1)},\quad \frac{5y}{2x}-3=\frac{y-5}{x}.$

(17) $(x+a)(y+b)-(x-a)(y-b)=2(a-b)^2,$
$x-y+2(a-b)=0.$

(18) $\frac{3}{x}-\frac{4}{y}=5;\ \frac{4}{x}-\frac{5}{y}=6.$

(19) $\frac{a}{x}+\frac{b}{y}=\frac{ac}{b};\ \frac{b}{x}+\frac{a}{y}=\frac{bc}{a}.$

(20) $(a+b)(x+y)-(a-b)(x-y)=a^2;$
$(a-b)(x+y)+(a+b)(x-y)=b^2.$

53.—Three simultaneous equations may be proposed for solution involving three unknowns, and the method to be adopted will be similar to those given above; but two eliminations will be generally required.

Ex. (1) Solve
$$\left.\begin{aligned} 2x-3y+4z&=4\\ 3x+5y-7z&=12\\ 5x-y-8z&=5\end{aligned}\right\}\quad\begin{aligned}&\ldots(1)\\&\ldots(2)\\&\ldots(3)\end{aligned}$$

Eliminate z between two pairs of these equations.

(1) $\times 2$ gives $4x-6y+8z=8,$
(3) is $5x-y-8z=5,$
$\therefore 9x-7y=13$ . . . (4)

Again, (1) $\times 7$ gives $14x-21y+28z=28,$
(2) $\times 4$ $12x+20y-28z=48,$
$\therefore 26x-y=76$ . . . (5)

Between (4) and (5) eliminate y.

(4) $\times 7$ gives $182x-7y=532.$
Subtract $9x-7y=13,$
$\therefore 173x=519,$
$x=3.$

Then (5) gives $78-y=76,$
$y=2,$

And (1) gives $6-6+4z=4,$
$z=1.$

Similarly, if four simultaneous equations are given involving four unknowns, one of the unknowns must be eliminated between three pairs of the equations; then a second between the resulting equations; and so on.

Exercise 28.

Solve—

(1) $x-3y-2z=1;$
$2x-3y+5z=-19;$
$5x+2y-z=12.$

(2) $3x-2y=5;$
$4x-3y+2z=11;$
$x-2y-5z=-7.$

(3) $x+y=1;$
$y+z=9;$
$z+x=5.$

(4) $2x-3y=3;$
$3y-4z=7;$
$4z-5x=2.$

(5) $x+\frac{y}{2}+\frac{z}{3}=6;$
$y+\frac{z}{2}+\frac{x}{3}=-1;$
$z+\frac{x}{2}+\frac{y}{3}=17.$

(6) $3x-4y+6z=1;$
$2x+2y-z=1;$
$7x-6y+7z=2.$

(7) $ax+by+cz=a;$
$ax-by-cz=b;$
$ax+cy+bz=c.$

(8) $\frac{1}{x}+\frac{2}{y}=5;$
$\frac{3}{y}-\frac{4}{z}=-6;$
$\frac{3}{z}-\frac{4}{x}=5.$

(9) $x+2y+3z=4;$
$3x-2y-v=6;$
$2x-3z-3v=6;$
$y-4v=15.$

(10) $x-2y=3;$
$y-2z=4;$
$z-2u=5;$
$u-2v=6;$
$v-2x=-3.$

(11) $$\frac{2x-y}{3}=\frac{3y+2z}{4}=\frac{x-y-z}{5}=4.$$

(12) $$\frac{x-y}{a}=\frac{y-z}{b}=\frac{z+x}{c}=\frac{x-a-b}{a+b+c}.$$

CHAPTER XIV.

PROBLEMS PRODUCING SIMULTANEOUS SIMPLE EQUATIONS.

54.—It is sometimes easier in the solution of problems to employ more than one letter to represent the quantities to be found, and often necessary to do so: but in all cases, if the problem be determinate, the conditions will be sufficient to give as many equations as there are unknown quantities employed.

Ex. (1) Six men and two women earn £7 13*s.* in six days; three men and eight women earn £5 14*s.* in four days: determine a man's and woman's wages per day.

Let x shillings be a man's daily wages,
and y „ „ woman's „

$$\begin{aligned} \text{Then } 6(6x+2y)&=153, \\ \text{and } 4(3x+8y)&=114. \\ \text{Or } 36x+12y&=153 \\ 12x+32y&=114 \\ 36x+96y&=342 \\ 36x+12y&=153 \\ 84y&=189 \\ y&=2\tfrac{1}{4}=2s.\ 3d. \\ 12x=114-32y&=114-72=42 \\ x&=3\tfrac{1}{2}=3s.\ 6d. \end{aligned}$$

Exercise 29.

(1) A bill of £2 3*s.* 6*d.* was paid in halfcrowns and florins, and three times the number of florins exceeded by 2 twice the number of halfcrowns. How many coins were there of each kind?

(2) If A give B 5 shillings, he will then have 6 shillings less than B; but if he receive 5 shillings from B, three

times his money will be a sovereign more than four times B's. How much has each?

(3) A fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator, and to $\frac{2}{7}$ if 3 is added to its denominator. Find it.

(4) A grocer sells 3 lbs. of tea and 5 lbs. of coffee for £1 0*s*. 11*d*., and 5 lbs. of tea and 3 lbs. of coffee for £1 6*s*. 5*d*. Find the prices per lb. of the tea and coffee.

(5) A person spends half-a-crown in apples and pears, buying his apples at 3 and pears at 2 a penny, and afterwards accommodates his neighbour with half his apples and a quarter of his pears at the same rate for 11½*d*. How many of each did he buy?

(6) Goods at £5 4*s*. and £3 5*s*. per ton are bought for £84 10*s*.; and when the former have advanced one-third and the latter one-fourth in price, they are sold for £108 17*s*. 6*d*. How much is there of each kind?

(7) A person having 12 miles to walk, proceeds at an uniform pace for the first 5 miles, and then at an increased but uniform pace for the remaining distance, taking 3 hours 25 minutes for the whole: had he quickened his pace 2 miles before, he would have saved 10 minutes. What were his rates of walking?

(8) The first digit of a number is, when doubled, 3 more than the second, and the number itself is less by 6 than five times the sum of its digits. Find the number.

(9) Find two fractions with numerators 2 and 5, whose sum is 1⅓; and if their denominators are interchanged, their sum is 2.

(10) If the sides of a rectangular field were each increased by two yards, the area would be increased by 220 yards; if the length were increased and the breadth diminished by 5 yards, the area would be diminished by 185 yards. What is its area?

(11) A train travels a certain distance in a certain time: had the rate been increased by 5 miles per hour, the time would have been 36 minutes less: had it been decreased by 4 miles per hour, the time would have been 36 minutes more. Find the distance and the time.

(12) A earns twice as much, and spends three times as much, as B, and at that rate incurs debt to the same amount as B saves: if A spent half of what he does, he would save 18 guineas in 96 days. What are the daily wages of each?

(13) A fraction which is equal to $\frac{2}{3}$ is increased to $\frac{8}{11}$ when a certain number is added to both its numerator and denominator, and diminished to $\frac{5}{9}$ when one more than the same number is subtracted from each. Find the fraction.

(14) A traveller (B) starts an hour after another (A), and overtakes him in 4 hours; but if A's rate per hour had been a mile more, B would have overtaken him only in 9 hours. What are their rates?

(15) 24 boxes and 20 bales will exactly fill a warehouse, and 6 boxes and 14 bales will fill half of it. How many boxes or bales alone would fill it?

(16) A number consists of three digits, of which the first and last are alike. By interchanging those in the units' and tens' places, the number is increased by 54; but if those in the tens' and hundreds' places are interchanged, 9 must be added to four times the result to make it equal to the original number. What is the number?

(17) Two workmen together can complete some work in 20 days; but if the first worked twice as fast, and the second half as fast, they would do it in 15 days. In how long a time could each do it alone?

(18) A person exchanged 12 bushels of wheat for 8 bushels of barley and £1 15*s.*, offering at the same time to sell a certain quantity of wheat for an equal quantity of barley and £2 10*s.* in money, or for £12 10*s.* in money. Determine the prices of the wheat and barley per bushel.

(19) A countryman being employed by a poulterer to drive a flock of geese and turkeys to London, in order to distinguish his own from any he might meet on the road, pulled 3 feathers out of the tail of each turkey, and 1 out of the tail of each goose, and upon counting them found that the number of turkey-feathers exceeded twice the goose-feathers by 15. He afterwards bought on the

way 10 geese and sold 15 turkeys, and found on his arrival that the number of turkeys was $\frac{3}{7}$ of the number of geese. How many of each had he at first?

(20) A bag contained halfcrowns, half-sovereigns, and sovereigns, amounting to £25 10s.: the number of coins at first was 78, but when the halfcrowns were changed for their value in sovereigns, the number was 29. How many half-sovereigns were there?

(21) A, B, C subscribed between them £20. If A's subscription had been one-tenth less, and B's one-tenth more, C's must have been increased by 8s. to make up the sum: but if A's had been one-eighth more, and B's one-eighth less, C's subscription would have been £3 10s. What did each subscribe?

(22) Round two wheels, whose circumferences are proportional to 5 and 3, two ropes are wound, whose difference exceeds the difference of the circumferences by 280 yards. The longer rope applied to the larger wheel winds round it a number of times greater by 12 than the shorter round the smaller wheel; and if the larger wheel turns round three times as fast as the smaller, the ropes will be unwound in the same time. Find the lengths of the ropes and the circumferences of the wheels.

(23) A belt of trees 4 yards apart is planted 5 rows in breadth round a rectangular field. If there were 4 rows down the longer sides, and 6 down the shorter, there would be 16 trees less; if 6 rows down the longer, and 3 down the shorter, 34 trees less. The trees being arranged so that one is placed at each corner, find the dimensions of the field.

(24) A certain sum was given away in charity to 14 men and 15 women. If there had been only 8 women, and the rest had been divided among the men, each man would have received twice as much as each woman; but if the sum had been less by 12s., and only half the number of men relieved, the rest when divided among the women would have allowed to each 2s. more than each man received. What sum was given away?

(25) A train carries first, second, and third class pas-

sengers, at 3*s.* 6*d.*, 2*s.* 6*d.*, 1*s.* 8*d.* fares. The number of first and third class passengers together is 2 less than three times the number of second class; the total fares paid by the first class amount to 6*s.* more than those paid by the second class; and the first and second class fares together are a shilling more than half as much again as the third class fares. How many passengers of each class are there?

(26) A train (A) starts from P towards Q, and an hour afterwards another train (B) starts from Q towards P, at a rate of 10 miles an hour more than A's: they meet half-way between P and Q. Had A started an hour after B, its rate must have been increased by 28 miles an hour in order to meet B at the same point. Find the length of the line.

(27) A pays to B and C as much as each of them has; B pays to A and C half as much as each of them then has; and C pays to A and B a third of what each of them then has. In the end A finds that he has 6*s.*, B 8*s.* 8*d.*, and C 2*s.* 4*d.* How much had each at first?

(28) A railway train, after travelling for an hour, meets with an accident which detains it half-an-hour, after which it proceeds at four-fifths of its usual rate, arriving in consequence an hour and a quarter late. If the accident had happened 30 miles farther on, the train would have arrived only an hour late. Determine the usual rate of the train.

(29) If the second and third of three vessels were filled from the first, there would remain in the first 2 gallons less than the third would hold; if the third were filled from the second, there would remain in the second a gallon less than a quarter of the first; and if the third were filled six times, and emptied into the first and second, there would remain in it the last time enough to fill a third part of the second vessel. What is the size of each?

(30) In a mile race, A gives B a start of 100 yards, and beats him by 15 seconds: when, in running the same distance again, A gives B a start of 45 seconds, he is beaten by 22 yards. Determine the rate of each in miles per hour.

MISCELLANEOUS EXAMPLES.

Exercise 30.

(1) If $a = 2, b = 3, x = 1$, find the value of
$$3a^3b^2x - abx^4 + 7ab^5 - 10a^5x.$$

(2) From the sum of
$$3x^4 - x^3y + 4x^2y^2 + y^4 \text{ and } 2x^3y - 5x^2y^2 - 2y^4$$
take the sum of
$$-x^4 - x^2y^2 - y^4 \text{ and } 4x^4 - x^3y + x^2y^2 - 3xy^3.$$

(3) Multiply $a^3 - 3a^2b - 5ab^2$ by $4a^2 - 7ab - b^2$.

(4) Divide $16a^4 + 4a^2b^2 + b^4$ by $4a^2 - 2ab + b^2$.

(5) a boys have x pence each; how many pence will they each have when y shillings are divided equally among them?

(6) Write down all the factors of a^2b; and all the common factors of $3x^3y$ and $4x^2y^2z$.

(7) Simplify (i) $3a - (x + 2a) - \{2x - 3(2a - x)\}$.

(ii) $a^2 - (a - b)(a - c)$.

(iii) $\dfrac{a(b^2 - c^2)}{bc} + \dfrac{2b(c^2 - a^2)}{ac} - \dfrac{c(a^2 + 2b^2)}{ab}$.

(8) Resolve into factors $x^2 - 4y^2$, $3x^3y - 3xy^3$, $x^2 - 7x - 18$.

(9) Solve the equations—

(i) $\dfrac{2x}{5} - \dfrac{7x}{9} - \dfrac{13}{15} + \dfrac{2x}{3} = 0$.

(ii) $\cdot25x - 1\cdot3 - 3x = \cdot005x + 4\cdot21$.

(iii) $\dfrac{3}{2x - 1} - \dfrac{1}{x - 2} = \dfrac{1}{2x}$.

(iv) $2(x - 1) = 3(y - 2)$; $\dfrac{x - 1}{2} = \dfrac{2y - 3}{3}$.

(10) Find the G. C. M. of $3x^3 - 2x^2 - 2x - 5$ and
$$3x^4 + 3x^3 + 2x^2 - x - 1.$$

(11) Half of a certain number of persons receive a shilling each, a third receive tenpence each, and the rest ninepence each. The whole sum received being £1 12*s.* 6*d.*; how many persons are there?

(12) Find the values of $3ab - 4ac + 5b^2$, and of $\frac{3a^2 - ac}{2b^2} - \frac{b^2 - c^2}{a^2}$ (1) when $a = 2$, $b = 3$, $c = 1$; (2) when $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = 0$.

(13) Subtract from $m^3 + n^3$ the sum of twice $m^2n - 3n^3$ and three times $m^3 - m^2n - mn^2$.

(14) Multiply $4a^3 - a^2b - 3ab^2$ by $2a^2b - 6ab^2 + 4b^3$.

(15) Divide $21x^5 + 4x^4y - 11x^3y^2 + xy^4$ by $7x^2 - xy - y^2$.

(16) Find the L. C. M. of $3a^3bc^2$, $4a^2b^3c$, $5c^6$, $6a^4c^2$.

Two expressions have $2a^3bxy$ for their G. C. M., and $24a^4b^3x^3y^2$ for their L. C. M., and one of the expressions is $6a^4bx^3y$: find the other.

(17) Simplify $\frac{2a - b}{ab} + \frac{3b - 2c}{bc} + \frac{4c - 3a}{ac}$; and

$$\frac{2x^3y}{15xy^2z} \div \frac{4xz^3}{9y^4z}.$$

(18) How many sheep worth c shillings apiece are equal in value to n oxen worth b pounds apiece?

(19) If $a = 5$, $b = 3$, $x = 4$, $y = 2$, find the values of

$$\frac{7a - 8b}{x + 3y} \text{ and } \frac{a^y \sim b^x}{y^a \sim x^b}.$$

(20) Add together $3a^3 - 2a^2x + 4ax^2 + x^3$, $-4a^3 + 3x^3$, $5a^2x - 6ax^2 - 2x^3$, $a^3 - 3a^2x - 4x^3$, and $-2a^3 + 2ax^2 - x^3$.

(21) From $3a^4 - 2a^2b^2 + 7ab^3 - b^4$ take $2a^4 + a^3b - ab^3 - 3b^4$.

(22) Multiply $4x^3y - 6x^2y^2 - 2xy^3$ by $2x^2 + 3xy - y^2$.

(23) Divide $9x^6 - 4x^4y^2 - 18x^3y^3 - 8x^2y^4 + 5y^6$ by $3x^3 - 2x^2y - 5y^3$.

(24) Write down the G. C. M. and L. C. M. of $28a^2bx^3y^3$ and $35a^3x^4y$.

(25) Simplify $\frac{2x - y}{xy} + \frac{2x^2 - y^2}{x^2y} + \frac{x^2 + y^2}{xy^2}$.

(26) A man who has £a buys b things at c shillings each, and sells x things at y pence each: express in pounds, and in guineas, the amount he has in the end.

(27) A certain number of guineas exceeds by a sovereign twice that number of shillings and thrice that number of crowns. What is the number?

(28) Divide $5x^6 - 6x^5 + 1$ by $x^2 - 2x + 1$.

) Reduce to its lowest terms the fraction

$$\frac{x^3 - 4x^2 + 5}{x^3 + 1}.$$

) Solve (i) $x - \frac{x-2}{3} = \frac{x+23}{4} - \frac{10+x}{5}.$

(ii) $\frac{7x+1}{x-1} = \frac{35}{9}\left(\frac{x+4}{x+2}\right) + \frac{28}{9}.$

) Find the value of the expression
$b-c).(a-b+c) + 2(a+b)^2 - 3(a^2-b^2)$ when $a=3$,
, $c=-6$; and simplify
$-2x^2y) + \{x^2y - 3y(xy-2y^2)\} - 5\{x^3 - (x^2y + 2xy^2 - y^3)\}.$

) Find the G. C. M. of
$3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21$:
the L. C. M. of
$6x^2y$, $10xy^2$, $4x(x+y)^2$, and $20y(x^2-y^2)$.

) Simplify $\frac{a - \frac{b^2}{a}}{(a+b)^2} - \frac{\frac{a^2}{b} - b}{(a-b)^2}$, and $\frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}$.

) Solve the equations—

(i) $(x-5)(x-2) - (x-5)(2x-5) + (x+7)(x-2) = 0.$

ii) $\begin{cases} \frac{3x+1}{15} - \frac{2x-8}{7y-32} = \frac{x-1}{5}. \\ 14x + 8y = -9. \end{cases}$

ii) $\begin{cases} 3x + y - 5z = 8 \\ 4x + 3y + 8z = 32 \\ 5x - 3y + 3z = 6. \end{cases}$

) What fraction is it which becomes 1 when 3 is ed to its numerator, and $\frac{1}{2}$ when 2 is added to its ominator?

) At what times between 2 and 3 o'clock will there difference of just five minutes between the positions he hour and minute hands of a watch?

) Resolve into factors $4a^3b - 4a^2b^2$, $4a^3b - 4ab^3$, $b + 24a^2b^2 - 36ab^3$.

) Obtain the L. C. M. of $6y^5z$, $5xy^4z^2$, $4x^2y^3z^3$, $3x^3yz^4$.

(39) Find the value of $\frac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$ when $a=4$, $b=\frac{1}{2}$, $c=1$.

(40) Show that the product of $x^2+2xy-3y^2$ and $x^2-5xy+4y^2$ is equal to that of $x^2-xy-12y^2$ and $(x-y)^2$.

(41) Obtain the product of the sum, product, and difference of the squares, of a and b; and write down the simple expressions of three dimensions that can be formed of a and b.

(42) If $m=6$, $n=4$, $x=3$, $y=1$, find the values of $2mx-4ny$, $2m^2n-3xy^2$, and $\frac{4x^n-3y^m}{2x-3y}$.

(43) Add together $3a^3b-4a^2b^2-2ab^3$, $5a^2b^2+ab^3+b^4$, and $a^4-3a^3b-2a^2b^2+ab^3$.

Also subtract the sum of the first two from the last.

(44) Multiply $ax^3-4a^2x^2-a^3x$ by $5a^2x^2-a^3x+3a^4$.

(45) Divide $12x^5y^2-23x^4y^3+7x^3y^4-9x^2y^5-2xy^6$ by $4x^2y-xy^2+2y^3$.

(46) From $\frac{5x^2-2y^2}{6xy}$ take $\frac{2x^2-5xy}{4y^2}$.

(47) Two men can do some work separately in a days and b days; what fraction of it will they do together in c days?

(48) Solve $\frac{ax}{b}+\frac{bx}{c}-\frac{cx}{a}=\frac{b}{c}(c+x)-\frac{a^2}{c}$.

(49) A bell tolled a certain number of strokes in a given time: if there had been 3 more strokes, the interval between the strokes would have been one second less; if two fewer, the interval would have been one second more. How many strokes were there, what was the interval between the strokes, and how long did the bell toll?

(50) Express in trinomial terms

$$x^2-2xz+z^2-y^2+2y-1,$$

and thence resolve it into factors.

(51) Find the value of $\frac{3a}{b}-\frac{4b}{x}+\frac{a-b}{x-y}-\frac{2ax}{by}$ when $a=7$, $b=3$, $x=6$, $y=4$.

) If £a are divided among b persons, how many ings will each receive?

) Add together $4a^3 - 2a^2c - 3ac^2$, $a^2c + ac^2 - c^3$, and $+ 3c^3 - a^3$; and from the sum subtract $-2a^3 + 2a^2c -$ $+ 2c^3$.

) Multiply $7x^3 - 2x^2y - 3xy^2$ by $4xy^2 - 6y^3$; and divide $-20a^2b + 2b^3$ by $a^2 - 3ab - b^2$.

) Simplify $a - (2b + 3a) - [4b + \{5a - (6b + 7a)\}]$, and $\{b + 3(a - 4b)\} + 5\{a - 6(b + 7a)\}$.

) Express with symbols, The sum of a and b to be nished by the difference of c and d (c being greater d), and the result to be multiplied by the sum of the res of e and f.

) Divide $(8a^3 - b^3) - (2a - b)^3$ by $(2a - b)^2 - (4a^2 - b^2)$.

) Separate into factors $12x - 36$, $x^2 - 9$, $x^2 + 5x + 6$, and $x - 6$; and write down their L. C. M.

) Find the G. C. M. of

$$20a^2 - 22ab + 6b^2 \text{ and } 12a^2 + 14ab - 10b^2.$$

) Simplify $\dfrac{x}{x + \dfrac{y}{y + \dfrac{z}{x}}}$ and $\dfrac{a + 1}{a + 1 + \dfrac{a - 1}{a - 1 + \dfrac{a}{a + 1}}}$

) A number of men earning each a shillings a day k with m men earning each b shillings a day, and the le amount of wages earned in c days is £mbc. How y men earn a shillings a day?

) Find the G. C. M. of $(a + b)^2 - c^2$ and $(b + c)^2 - a^2$.

) Divide the difference of the squares of $a^2 + bc$ and ac by $a + b - c$.

) Simplify $(3x - y)(x - 3y) - 3(x - y)^2 + (2x + y)^2$.

) Find the greatest common measure of $24a^3bc^2$, $36a^2b^2c^2$, $42a^4c$; and of $18x^3 - 3x^2y - 3xy^2$ and $24x^4 + 14x^3y + 2x^2y^2$.

) Simplify $\dfrac{a}{(a-b)(a-c)} - \dfrac{b}{(b-c)(b-a)} + \dfrac{c}{(c-a)(c-b)}$

$\dfrac{x^3y - xy^3}{(x-y)^3} \div \dfrac{x^3y^3}{x^3 - y^3}$.

[illegible]

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$$[illegible]$$

[illegible]

Divide a into two parts such that the square of one may be one-fourth of the square of their difference.

A person has 55 coins, consisting of crowns and gs, and their amount is £7 3s. How many has he 1 kind?

a inches added to b feet are equal to c yards, and a e half of c yards: express b in terms of c.

Square x^3+x^2+x+1, and divide the result by the of x^2+1.

Simplify $(2a-b)-2(a-b)+\{a-2(b-2a)\}$, and $(x-5)-(x-3)^2-x(x-3)$.

Resolve into factors x^2-2x-3, x^2-1, and $2x^2-6x$; rite down their L. C. M.

Find the G.C.M. of $8a^2+2ab-3b^2$ and
$$12a^3+10a^2b-4b^3.$$

Simplify $\dfrac{(x+y)^2-z^2}{x^2-(y-z)^2}$ and
$$\frac{x-1}{(x-2)(x-3)}-\frac{2(x-2)}{(x-1)(x-3)}+\frac{x-3}{(x-1)(x-2)}.$$

Solve—

(i) $\dfrac{x-6}{4(x-2)}=\dfrac{x-10}{6(x-2)}+\frac{1}{9}$.

(ii) $x-a-\dfrac{ab}{x-b}=\dfrac{x^2+ab}{x-b}$.

(iii) $\left.\begin{array}{l}3x-y-z=5\\3y-x-z=-7\\3z-x-y=5\end{array}\right\}$

Find a number of three digits, of which the middle equal to the sum of the other two, such that when st and third are interchanged the number is in- l by 99; when the first and second, by 180.

If two numbers are taken whose product is 1, show ie difference of the squares of their sum and differ- s 4.

Find the value of $a^3-b^3-c^3+d^3-3abc+3bcd-4ab^2$ $a=\cdot1$, $b=-\cdot2$, $c=\cdot3$, $d=0$.

A travelled a third of a given distance at 4 miles an and the rest at 12; B started a quarter of an hour

(67) Solve the equations—

(i) $\frac{x}{2}-\frac{2x}{3}+\frac{3x}{4}-\frac{4x}{5}+\frac{2(x-5)}{9}=0.$

(ii) $\frac{3-3x}{6+2x}=\frac{1}{2}-\frac{2(1-x)}{4-x}.$

(iii) $\left.\begin{array}{l} 7(x-1)=3(y+8) \\ \frac{4x+2}{9}=\frac{5y+9}{2} \end{array}\right\}$

(68) A person pays income-tax at the rate of $9d.$ in the pound, and if his income were increased by £10, and the tax were raised to 1*s.* in the pound, his net income would remain the same. Find his income.

(69) If 1 be divided into any two parts, show that their product subtracted from either part will leave the square of that part.

(70) Simplify—

$$(a+b)-\left[(2a-3b)-\{(5a+6b)-(-7a+b)\}\right].$$

(71) Find the value of the expression

$$\frac{x(x-1)x^3+(x^3+2x-2)a^2+(3x^3-x^2)a-x^4}{2a^2x+5a-x^2}$$

when $x=2$, $a=3$.

(72) Find the continued product of

$$x-a,\ x+a,\ x^2-ax+a^2, \text{ and } x^2+ax+a^2.$$

(73) Divide $abx^3+(ac-bd)x^2-(ae+cd)x+de$ by $ax-d$.

(74) Find the G. C. M. of

$$3x^4-10x^3+9x^2-2x, \text{ and } 2x^4-7x^3+2x^2+8x.$$

(75) Simplify—

$$\frac{x-a}{x-b}+\frac{x-b}{x-a}-\frac{(a-b)^2}{(x-a)(x-b)}; \text{ and } \frac{\frac{3}{4}(1+\frac{8}{5}x)}{1\frac{1}{2}-\frac{3}{5}\left(1-\frac{3x}{2}\right)}.$$

(76) Solve the equations—

(i) $15x-\frac{1}{2}(5x-1\frac{1}{2})=9-\frac{1}{4}(25x-7).$

(ii) $\frac{12x+1}{13x-8}+\frac{20x-13}{18}+\frac{2-10x}{9}=0.$

(iii) $\frac{1}{x+3}+\frac{2}{x+6}=\frac{3}{x+9}.$

(iv) $\left.\begin{array}{l} 2x-5=\frac{3}{7}(2y-1) \\ 16y-3=\frac{2}{5}(x+5) \end{array}\right\}$

(77) Divide a into two parts such that the square of one part may be one-fourth of the square of their difference.

(78) A person has 55 coins, consisting of crowns and shillings, and their amount is £7 3*s*. How many has he of each kind?

(79) a inches added to b feet are equal to c yards, and a feet are half of c yards: express b in terms of c.

(80) Square x^3+x^2+x+1, and divide the result by the square of x^2+1.

(81) Simplify $(2a-b)-2(a-b)+\{a-2(b-2a)\}$, and $(x-4)(x-5)-(x-3)^2-x(x-3)$.

(82) Resolve into factors x^2-2x-3, x^2-1, and $2x^2-6x$; and write down their L. C. M.

(83) Find the G. C. M. of $8a^2+2ab-3b^2$ and $12a^3+10a^2b-4b^3$.

(84) Simplify $\dfrac{(x+y)^2-z^2}{x^2-(y-z)^2}$ and

$$\frac{x-1}{(x-2)(x-3)}-\frac{2(x-2)}{(x-1)(x-3)}+\frac{x-3}{(x-1)(x-2)}.$$

(85) Solve—

(i) $\dfrac{x-6}{4(x-2)}=\dfrac{x-10}{6(x-2)}+\dfrac{1}{9}$.

(ii) $x-a-\dfrac{ab}{x-b}=\dfrac{x^2+ab}{x-b}$.

(iii) $\left.\begin{aligned}3x-y-z&=5\\3y-x-z&=-7\\3z-x-y&=5\end{aligned}\right\}$

(86) Find a number of three digits, of which the middle one is equal to the sum of the other two, such that when the first and third are interchanged the number is increased by 99; when the first and second, by 180.

(87) If two numbers are taken whose product is 1, show that the difference of the squares of their sum and difference is 4.

(88) Find the value of $a^3-b^3-c^3+d^3-3abc+3bcd-4ab^2$ when $a=\cdot1$, $b=-\cdot2$, $c=\cdot3$, $d=0$.

(89) A travelled a third of a given distance at 4 miles an hour, and the rest at 12; B started a quarter of an hour

later and travelling at the rate of 10 miles an hour reached the end 13′ sooner. Find the distance.

(90) Simplify and resolve into factors—

$$(x^3-2x^2y)-(3xy^2+y^3)-\{x^3-(xy^2+y^3)\}, \text{ and}$$
$$3(x+y)^2-\{5(x+y)(x-y)+12xy\}+7(x-y)^2.$$

(91) Find the G. C. M. and L. C. M. of

$$2a^2b(12a^2-11ab+2b^2) \text{ and } 3ab^2(20a^2+7ab-3b^2).$$

(92) Simplify $\dfrac{(p-q)^2-(q-r)^2}{q(p^2-r^2)}$, and

$$\frac{p-q}{(p-2q)(p-3q)}-\frac{p-3q}{(p-q)(p-2q)}.$$

(93) Solve—

(i) $\dfrac{x}{x-1}+\dfrac{2}{x-2}=\dfrac{x}{x-3}$.

(ii) $\left.\begin{array}{l}3x-2y=1\\5x+7y=-6\end{array}\right\}$

(94) The two longest sides of a triangle exceed the shortest by 12, and the two shortest exceed the longest by 2: find the middle side.

(95) In the product of $5x^4-3x^3+4x^2-2x+1$ and $x^4+2x^3+3x^2+4x+5$, find the coefficient of x^4.

(96) Show that $(x-y)^3+3(x-y)^2+3(x-y)+1$ and $x^3-3x^2(y-1)+3x(y-1)^2-(y-1)^3$ are identical.

(97) The product of two expressions is

$$8q^3+r^3-6pqr-2p^2q-p^2r,$$

and one of them is $4q^2+r^2-2qr-pr-2pq$: find the other.

(98) If $a+b+c=2s$, show that $s(s-c)+(s-a)(s-b)=ab$, and that $s^2-(s-a)^2-(s-b)^2+(s-c)^2=2ab$.

(99) Solve $ax+by+cz=bx+cy+az=cx+ay+bz=1$.

(100) If x^2+ax+b and x^2+px+q have a common factor, find the other factor in each expression; and show that

$$\frac{b-q}{a-p}=\frac{pb-aq}{b-q}.$$

CHAPTER XV.

INVOLUTION AND EVOLUTION.

55.—The operation of raising an expression to any required power (5) is called *Involution*, and may be performed by the ordinary rules of multiplication.

Thus the square of $3a^4$ is $3a^4 \times 3a^4 = 9a^8$; the cube of $a+b$ is $(a+b)\times(a+b)\times(a+b) = a^3+3a^2b+3ab^2+b^3$.

56.—In raising a *simple expression* to any required power, it will be seen that

The numerical coefficient must be raised to that power as in ordinary arithmetic; but

The index of any letter must be multiplied by the index of the power.

For in the first of the above examples, which may be written $(3a^4)^2$, the co-efficient 9 is 3-squared, but the index 8 is $4+4$, *i.e.* 4 multiplied by 2.

Similarly $(2x^3)^4$ is $16x^{3+3+3+3}$, *i.e.* $16x^{12}$.

Expressed in a general form, we shall have, as the proof of this rule for indices,

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots n \text{ factors},\\ &= a^{m+m+m+\ldots n \text{ terms}},\\ &= a^{mn}.\end{aligned}$$

57.—All powers of a *positive* quantity will be positive; but of a *negative* quantity, the even powers will be positive, and the odd negative.

For
$$\begin{aligned}(-a)^2 &= -a \times -a = +a^2,\\ (-a)^3 &= +a^2 \times -a = -a^3,\\ (-a)^4 &= -a^3 \times -a = +a^4,\\ &\text{\&c.}\end{aligned}$$

It may be observed also that two compound expressions whose terms are alike but have opposite signs will have their even powers identically alike.

Thus
$$\begin{aligned}(b-a)^2 &= \{-(a-b)\}^2 = (a-b)^2,\\ (xy-x^2-y^2)^4 &= \{-(x^2-xy+y^2)\}^4 = (x^2-xy+y^2)^4.\end{aligned}$$

58.—The process of raising a compound expression to a high power may be shortened by noticing that two powers

multiplied together will give as their result a power whose index is the *sum* of their indices: the third power, for instance, when multiplied by the second, will give the fifth; the cube, when squared, will give the sixth; &c.

The form of the *square* of a binomial has been already noticed (29. i, ii); the *cube* of a binomial may also be remembered with advantage; viz.

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

If the second term be negative, its odd powers in the result will be negative (57),

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

Thus
$$(x+2y)^3=x^3+3x^2(2y)+3x(2y)^2+(2y)^3,$$
$$=x^3+6x^2y+2xy^2+8y^3.$$

Exercise 31.

(1) Write down the squares and cubes of a^3, x^5, x^2y^3, $2a^2bc^3$, $-5ax^3y^2$, $7m^3nx^2y^4$.

(2) Write down the fifth and sixth powers of $2a^2b$, $-3xy^3$, $-5a^2bx^3$, $\frac{a^3b^2}{2}$, $-\frac{2x^3y}{3abc}$.

Expand—

(3) $(a+x)^3$, $(2x-y)^4$, $(x^2y-2xy^2)^6$, $(ab-3)^7$.

(4) $(1-a-a^2)^2$, $(2-3x+4x^2)^3$, $(a^2+xy+y^2)^5$.

N.B.—The form of the square of a multinomial expression given in Ex. 16 (10) should be noticed.

EVOLUTION. SIMPLE EXPRESSIONS.

59.—The process reverse of Involution is called *Evolution*, and has this object:—An expression being given, to find another whose square, or cube, or fourth power, &c. may be equal to it.

The new expression is called a *root*, and is indicated as the *square* root, *cube* root, *fourth* root, &c. according as the given expression is to be its square, cube, fourth power, &c.

The symbol denoting a square root to be extracted is $\sqrt{}$;

for other roots the same symbol is employed, but with a figure prefixed above to indicate the root, thus, $\sqrt[3]{}$ $\sqrt[4]{}$. When the expression consists of more than one letter or number, a bar is added over the top to show how far the symbol extends; as $\sqrt[3]{8a^3b^6}$, $\sqrt{a^2+2ab+b^2}$.

60.—The square root of a simple expression will be found by taking the arithmetical square root of the coefficient, and halving the indices: thus of $25a^2x^4$ the square root is $5ax^2$.

Any other root of a simple expression must be found in a similar way, the indices being divided by 3, 4, 5, . . . for the cube root, fourth root, fifth root, &c.

It follows from (57) that a negative quantity has no possible square root; and that a positive quantity has two square roots, equal but of opposite signs. The square root of 9, for example, is $+3$ or -3; $\sqrt{25a^2x^4}$ is $\pm 5ax^2$.

A root which is required but cannot be exactly obtained is called a *Surd*; such as $\sqrt{3}$, $\sqrt{a}$, $\sqrt[3]{x^2y}$.

Exercise 32.

(1) Extract the square roots of a^4, x^8, $4a^6b^2$, $2x^4y^6$, $49a^2x^6y^{10}$, $100m^2n^4x^6y^{10}z^8$.

(2) Simplify $\sqrt[3]{64}$, $\sqrt[4]{x^8}$, $\sqrt[5]{a^5x^{15}y^{10}}$, $\sqrt[4]{16a^{12}b^4c^8}$, $\sqrt{144c^6d^4x^2y^8}$.

(3) Simplify $\sqrt{25a^2b^4c^2}+\sqrt[3]{8a^3b^6c^3}-\sqrt[4]{81a^4b^8c^4}-\sqrt[5]{32a^5b^{10}c^5}$.

(4) Simplify $\sqrt[3]{27x^3y^6}\times\sqrt[5]{243y^5z^5}\times\sqrt{16x^4z^2}$.

Find the value when $a=1$, $b=3$, $x=2$, $y=6$,

(5) Of $4\sqrt{2x}-\sqrt{abxy}+5\sqrt{a^2b^3xy}$.

(6) Of $2a\sqrt{8ax}+b\sqrt[3]{12by}+4abx\sqrt{bxy}$.

SQUARE ROOT OF COMPOUND EXPRESSIONS.

61.—A process for finding the square root of a compound expression may be obtained by analysing the *square* of a binomial.

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We know that the square of $a+b$ is $a^2+2ab+b^2$: our problem here is—Supposing that $a^2+2ab+b^2$ were given, how may we obtain from it in a complete form its root $a+b$?

The first term a is obviously the square root of the a^2;

If subtracted, the remainder is $2ab+b^2$ or $(2a+b)b$, and the second term b will be the exact quotient when this remainder is divided by $2a+b$.

How is this divisor $2a+b$ found? By *doubling the part of the root already found and adding the new term* (b) *itself.*

The arrangement of the work may be made as follows:—

$$\begin{array}{rl} & a^2+2ab+b^2\ (a+b \\ & \underline{a^2} \\ 2a+b) & 2ab+b^2 \\ & \underline{2ab+b^2}. \end{array}$$

Ex. 1. To find the square root of $25x^2-20x^3y+4x^4y^2$.

$$\begin{array}{rl} & 25x^2-20x^3y+4x^4y^2\ (5x-2x^2y.\quad \textit{Ans.} \\ & \underline{25x^2} \\ 10x-2x^2y) & -20x^3y+4x^4y^2 \\ & \underline{-20x^3y+4x^4y^2}. \end{array}$$

The second term $-2x^2y$ being obtained by dividing $-20x^3y$ by $10x$, and then annexed both to the root and the divisor.

It is clear that there will be a remainder unless the three terms form a perfect square.

The same process may be extended to longer expressions by considering the a to represent at each step *the part of the root already found*; each successive divisor being then the double of this + the new term.

Ex. 2. Extract the square root of

$$9x^4-12x^3+10x^2-4x+1.$$

$$\begin{array}{rl} & 9x^4-12x^3+10x^2-4x+1\ (3x^2-2x+1.\quad \textit{Ans.} \\ & \underline{9x^4} \\ 6x^2-2x) & -12x^3+10x^2 \\ & \underline{-12x^3+\ \ 4x^2} \\ 6x^2-4x+1) & 6x^2-4x+1 \\ & \underline{6x^2-4x+1}. \end{array}$$

Note.—Instead of doubling the root, it is obvious that

each divisor may be obtained from the preceding one by doubling its last term.

62.—It may be instructive to notice that the square of a multinomial, referred to in (58, *note*), may also be written as

$$a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d+ \text{\&c.}$$

where the quantities in the brackets are the successive divisors in the above process; and the expression taken to the end of either of its compound terms is a complete square.

Exercise 33.

Extract the square roots of—

(1) $9a^2+12ab+4b^2$. (2) $25a^2b^2-20ab^2c+4b^2c^2$.

(3) $a^4+4a^3+2a^2-4a+1$.

(4) $x^4-2x^3y+3x^2y^2-2xy^3+y^4$.

(5) $4a^6-12a^5x+5a^4x^2+6a^3x^3+a^2x^4$.

(6) $16p^4-32p^3q+24p^2q^2-8pq^3+q^4$.

(7) $9x^6-24x^4y^2-12x^3y^3+16x^2y^4+16xy^5+4y^6$.

(8) $4a^8+16a^6c^2-32a^2c^6+16c^8$.

(9) $m^8-4m^7+10m^6-20m^5+35m^4-44m^3+46m^2-40m+25$.

(10) $25x^6-30x^5y-31x^4y^2+34x^3y^3+10x^2y^4-8xy^5+y^6$.

(11) $a^6+b^6+6ab(a^4+b^4)+15a^2b^2(a^2+b^2)+20a^3b^3$.

(12) $x^4-x^3y-\frac{7}{4}x^2y^2+xy^3+y^4$.

(13) $x^4-4x^3y+6x^2y^2-6xy^3+5y^4-\frac{2y^5}{x}+\frac{y^6}{x^2}$.

(14) $\frac{a^4}{9}-\frac{a^3x}{2}+\frac{43}{48}a^2x^2-\frac{3}{4}ax^3+\frac{x^4}{4}$.

(15) $1+\frac{4}{x}+\frac{10}{x^2}+\frac{20}{x^3}+\frac{25}{x^4}+\frac{24}{x^5}+\frac{16}{x^6}$.

(16) $x^4+2(a-b)x^3+(a^2+b^2)x^2+2ab(a-b)x+a^2b^2$.

(17) $\frac{a^2}{b^2}-\frac{2a}{b}+3-\frac{2b}{a}+\frac{b^2}{a^2}$.

(18) $1+a$ to five terms.

We know that the square of $a+b$ is $a^2+2ab+b^2$: our problem here is—Supposing that $a^2+2ab+b^2$ were given, how may we obtain from it in a complete form its root $a+b$?

The first term a is obviously the square root of the a^2;

If subtracted, the remainder is $2ab+b^2$ or $(2a+b)b$, and the second term b will be the exact quotient when this remainder is divided by $2a+b$.

How is this divisor $2a+b$ found? By *doubling the part of the root already found and adding the new term* (b) *itself.*

The arrangement of the work may be made as follows:—

$$\begin{array}{rl} & a^2+2ab+b^2\ (a+b \\ & \underline{a^2} \\ 2a+b) & 2ab+b^2 \\ & \underline{2ab+b^2}. \end{array}$$

Ex. 1. To find the square root of $25x^2-20x^3y+4x^4y^2$.

$$\begin{array}{rl} & 25x^2-20x^3y+4x^4y^2\ (5x-2x^2y.\quad \textit{Ans.} \\ & \underline{25x^2} \\ 10x-2x^2y) & -20x^3y+4x^4y^2 \\ & \underline{-20x^3y+4x^4y^2}. \end{array}$$

The second term $-2x^2y$ being obtained by dividing $-20x^3y$ by $10x$, and then annexed both to the root and the divisor.

It is clear that there will be a remainder unless the three terms form a perfect square.

The same process may be extended to longer expressions by considering the a to represent at each step *the part of the root already found*; each successive divisor being then the double of this + the new term.

Ex. 2. Extract the square root of

$$9x^4-12x^3+10x^2-4x+1.$$

$$\begin{array}{rl} & 9x^4-12x^3+10x^2-4x+1\ (3x^2-2x+1.\quad \textit{Ans.} \\ & \underline{9x^4} \\ 6x^2-2x) & -12x^3+10x^2 \\ & \underline{-12x^3+\ \ 4x^2} \\ 6x^2-4x+1) & 6x^2-4x+1 \\ & \underline{6x^2-4x+1}. \end{array}$$

Note.—Instead of doubling the root, it is obvious—

Ex. 1. To find the cube root of $8x^6+60x^5y+150x^4y^2+125x^3y^3$.

$$\begin{array}{rl|l}
 & & 8x^6+60x^5y+150x^4y^2+125x^3y^3\,(2x^2+5xy. \quad \textit{Ans.} \\
 & & 8x^6 \\ \hline
3(2x^2)^2 = & 12x^4 & 60x^5y+150x^4y^2+125x^3y^3 \\
3\times 2x^2\times 5xy = & \quad 30x^3y & \\
(5xy)^2 = & \qquad 25x^2y^2 & \\ \cline{2-2}
 & 12x^4+30x^3y+25x^2y^2 & \underline{60x^5y+150x^4y^2+125x^3y^3}.
\end{array}$$

The same method may be applied to longer expressions, by considering the a in the typical form $3a^2+3ab+b^2$ to represent at each stage of the process *the part of the root already found.*

Ex. 2. Extract the cube root of $8x^6-12x^5y+54x^4y^2-49x^3y^3+108x^2y^4-48xy^5+64y^6$.

$$\begin{array}{rl|l}
 & & 8x^6-12x^5y+54x^4y^2-49x^3y^3+108x^2y^4-48xy^5+64y^6\,(2x^2-xy+4y^2 \\
 & & 8x^6 \\ \hline
3(2x^2)^2 = & 12x^4 & -12x^5y+54x^4y^2-49x^3y^3 \\
3\times 2x^2\times -xy = & \quad -6x^3y & \\
(-xy)^2 = & \qquad +x^2y^2 & \\ \cline{2-2}
 & 12x^4-6x^3y+x^2y^2 & \underline{-12x^5y+6x^4y^2-x^3y^3} \\
3(2x^2-xy)^2 = & 12x^4-12x^3y+3x^2y^2 & 48x^4y^2-48x^3y^3+108x^2y^4-48xy^5+64y^6 \\
3(2x^2-xy)4y^2 = & \qquad 24x^2y^2-12xy^3 & \\
(4y^2)^2 = & \qquad\qquad +16y^4 & \\ \cline{2-2}
 & 12x^4-12x^3y+27x^2y^2-12xy^3+16y^4 & \underline{48x^4y^2-48x^3y^3+108x^2y^4-48xy^5+64y^6}.
\end{array}$$

N.B.—If $a+b$ are the first two terms of the root, the first line (a) of the second division is

$$3(a+b)^2 \text{ or } 3a^2+6ab+3b^2.$$

This may also be obtained by adding to the preceding divisor its second term and twice its third term,

$$\begin{array}{r} 3a^2+3ab+b^2 \\ +3ab+2b^2 \\ \hline 3a^2+6ab+3b^2, \end{array}$$

a method which in long arithmetical questions will materially shorten the work.

65.—The method employed in arithmetic is identical with the above.

Ex. To find the cube root of 15625.

$$\begin{array}{rr|l}
 & & 15625\ (25 \\
2^3 = & & 8 \\
3a^2 = \quad 3(20)^2 \quad = & 1200 & 7625 \\
3ab = 3\times20\times5 = & 300 & \\
b^2 = \quad 5^2 \quad = & 25 & \\
\hline
 & 1525 & 7625
\end{array}$$

a being $=20$, and $b=5$.

In the divisor a change may be conveniently made by transferring the cyphers of the square to the 3; *i.e.* instead of $3(20)^2$ take 300×2^2; and, similarly, instead of $3\times20\times5$ take $30\times2\times5$.

66.—As in (62) the *cube* of a multinomial $(a+b+c+\ldots.)$ may be written as

$$a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c+\ldots.$$

where the quantities within the brackets are the successive divisors of the above process; and the expression taken to the end of either of its compound terms is a complete cube.

Exercise 34.

Extract the cube root of—

(1) $a^3+6a^2b+12ab^2+8b^3$. (2) $8a^3+12a^2x+6ax^2+x^3$.

(3) $x^3y^3+12x^2y^4+48xy^5+64y^6$.

(4) $8b^3-60b^2y+150by^2-125y^3$.

(5) $27a^6+54a^4bc+36a^2b^2c^2+8b^3c^3$.

(6) $x^6+2x^5y+6x^4y^2+7x^3y^3+6x^2y^4+3xy^5+y^6$.

(7) $a^6-6a^5+9a^4+4a^3-9a^2-6a-1$.
(8) $64x^6+192x^5+144x^4-32x^3-36x^2+12x-1$.
(9) $1-9x+39x^2-99x^3+156x^4-144x^5+64x^6$.
(10) $x^6+9x^5y-135x^3y^3+729xy^5-729y^6$.
(11) $1-3x+6x^2-10x^3+12x^4-12x^5+10x^6-6x^7+3x^8-x^9$.
(12) $1+x$ to four terms.

67.—The fourth power being the square of the square, the fourth root will be the square root of the square root.

And the sixth power being the square of the cube, the sixth root will be the cube root of the square root. Similarly the eighth, ninth, twelfth, &c. roots could be found.

A method of the same kind as that given above for cube root might be devised for extracting the fifth or seventh root, but it would not be practically useful.

Exercise 35.

(1) Find the fourth root of
$81a^4-540a^3b+1350a^2b^2-1500ab^3+625b^4$.
(2) The sixth root of
$64-192x+240x^2-160x^3+60x^4-12x^5+x^6$.
(3) The eighth root of
$1-8y^2+28y^4-56y^6+70y^8-56y^{10}+28y^{12}-8y^{14}+y^{16}$.

CHAPTER XVI.

QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY.

68.—An equation which involves the square of the unknown quantity, but no higher power, is called *quadratic*. If it involves only the square, it is said to be a *pure* quadratic equation; but if it involves the simple power also, it is said to be *adfected*.

Thus $3x^2-12=0$ is pure; but $3x^2-2x=8$ is adfected.

69.—A pure quadratic (in x) must be reduced in the same manner as a simple equation until x^2 is found; and then the square root of each side must be taken.

Ex. Solve $(x-3)(x+5)=2x+1$.

$$\therefore\ x^2+2x-15=2x+1,$$
$$\therefore\ x^2=16,$$
$$\text{and } x=\pm 4.$$

It will be noticed that there are *two* roots (60) equal in magnitude, but of opposite signs.

70.—The solution of an adfected quadratic depends upon this, that an expression of two terms, such as x^2+6x, in which the first term is positive, and has unity for coefficient, can be made a complete square by *annexing, as a third term, the square of half the coefficient of the second.*

Thus x^2+6x becomes a square by adding 3^2; x^2+6x+9 being then the square of $x+3$.

So x^2-7x becomes $\left(x-\frac{7}{2}\right)^2$ by adding $\frac{49}{4}$;

x^2+px becomes $\left(x+\frac{p}{2}\right)^2$ by adding $\frac{p^2}{4}$.

See Exercise 13 (5).

The application of the above to quadratic equations will be readily seen by an example:—

If $\qquad x^2+6x \quad =40,$

then, adding 9, $\qquad x^2+6x+9=49.$

Extract the square root of each side,

$$x+3=\pm 7.$$

$$\therefore x=-3+7 \text{ or } -3-7,$$

$$\text{i.e. } 4 \text{ or } -10,$$

either of which values will be found to satisfy the given equation.

If after the simplification of an equation and its arrangement by powers of x, x^2 have a coefficient, the whole must be divided out by this; and if x^2 be negative, the signs of all the terms must be changed, so as to make this positive.

Ex. (2) Solve $\dfrac{6}{x+1}+\dfrac{2}{x}=3.$

Simplifying (as in simple equations), we have

$$-3x^2+5x=-2.$$

Dividing by 3, and changing the signs,

$$x^2-\frac{5}{3}x=\frac{2}{3}.$$

Complete the square,

$$x^2-\frac{5}{3}x+\left(\frac{5}{6}\right)^2=\frac{2}{3}+\frac{25}{36}$$

$$=\frac{49}{36}.$$

$$\therefore x-\frac{5}{6}=\pm\frac{7}{6},$$

$$x=\frac{5\pm 7}{6}=2 \text{ or } -\frac{1}{3}.$$

Note.—On extracting the square root, each side of the equation might have a double sign prefixed; but if this is done, it will readily be seen that no different result will be obtained.

Ex. (3) $\qquad 9x^2+6x=-1.$

$$\therefore x^2+\frac{2}{3}x=-\frac{1}{9}.$$

Completing the square,

$$x^2+\frac{2}{3}x+\frac{1}{9}=0,$$

$$x+\frac{1}{3}=0,$$

$$x=-\frac{1}{3}.$$

Since, after completing the square, the right-hand side of the equation—on whose double sign the two different values of the root depend—here becomes 0, there is only *one* value to the root; but it is right to regard the equation as still having *two* roots, but *equal to one another*.

Equations, whether pure or adfected, may sometimes have *surd* roots (60), as in the following examples. Sometimes the number on the right-hand side, whose square root is to be taken, is negative, in which case the roots of the equation are *impossible* (60).

Ex. (4)
$$\frac{x^2-5}{3}+\frac{2x^2+1}{6}=\frac{1}{2},$$
$$2x^2-10+2x^2+1=3,$$
$$4x^2=12,$$
$$x^2=3,$$
$$x=\pm\sqrt{3}.$$

Ex. (5)
$$5x^2-12x=2,$$
$$x^2-\frac{12}{5}x=\frac{2}{5},$$
$$x^2-\frac{12}{5}x+\left(\frac{6}{5}\right)^2=\frac{36}{25}+\frac{2}{5},$$
$$=\frac{46}{25},$$
$$x-\frac{6}{5}=\pm\frac{\sqrt{46}}{5},$$
$$x=\frac{1}{5}(6\pm\sqrt{46}).$$

71.—*Solution by Inspection.*—A quadratic equation which has been reduced to its simplest form, and written with its terms entirely on one side, may sometimes have that side resolved by inspection into two factors; in which case the roots may be written down at once, without completing the square.

Ex. (6)
$$x^2-7x+12=0$$
may be written as $(x-3)(x-4)=0$.

Now it will be noticed that, if *either* of the factors on the left side is 0, the product of the two factors will be 0, and the equation will be satisfied.

So that $x-3=0$ and $x-4=0$ give the roots; *i.e.* $x=3$ or 4.

This is most easily applicable when no term occurs except those involving x. Thus

Ex. (7) $$5x^2+7x=0$$
is $x(5x+7)=0$,
and is satisfied if $x=0$, or if $5x+7=0$;

$\therefore$ the roots are 0 and $-\frac{7}{5}$.

An equation involving a higher power of x may be sometimes easily solved by this method.

Ex. (8) $$2x^3-x^2-10x=0,$$
or $x(2x^2-x-10)=0$;
$\therefore$ $x=0$; or else $2x^2-x-10=0$.

Solving $2x^2-x-10=0$ as a quadratic, we find two roots, -2 and $\frac{5}{2}$;

$\therefore$ the equation has *three* roots, 0, -2, $2\frac{1}{2}$.

Sometimes one root of an equation is observable at once; in that case, if a represent the root, $x-a$ will be found to be a factor of the equation (all the terms being written on one side); and the other roots may then be obtained as above.

Ex. (9) $$x^3-2x^2-11x+12=0.$$

Here, if $x=1$, the equation will be satisfied; and it will be found that the equation may then be written
$$(x-1)(x^2-x-12)=0;$$
And x^2-x-12 will $=0$
if $x=-3$ or 4,
$\therefore$ the three roots are 1, -3, 4.

Exercise 36.

Solve

(1) $3x^2-20=55$. (2) $2(x^2-1)-3(x^2+1)+14=0$.

(3) $(x+1)(x-2)+(x-1)(x+2)=4$.

(4) $\dfrac{x-7}{3}+\dfrac{2}{x+7}+2=\dfrac{7x-1}{24}$.

(5) $x^2+4x=12$. (6) $x^2-6x=16$.

(7) $x^2-7x=8$. (8) $x^2-12x+6=\dfrac{1}{4}$.

(9) $x^2-x=6$. (10) $x^2-\frac{2}{3}x+\frac{1}{12}=0$.

(11) $3x^2-4x=7$.

(12) $\frac{x^2}{2}-\frac{x}{3}=2(x+2)$.

(13) $5x^2-3x=2$.

(14) $12x^2+x-1=0$.

(15) $2x^2-27x-14=0$.

(16) $5x^2-8x=0$.

(17) $\frac{3x}{4}+\frac{4}{3x}=\frac{13}{6}$.

(18) $x(x+1)-3(x+2)=2$.

(19) $5x(x-3)-2(x^2-6)=(x+3)(x+4)$.

(20) $\frac{1}{7}(x-4)-\frac{2}{5}(x-2)=\frac{1}{x}(2x+3)$.

(21) $x^2+ax=0$.

(22) $ax^2-bx=0$.

(23) $\frac{2}{x-1}=\frac{3}{x-2}+\frac{2}{x-4}$.

(24) $(x-1)(x-2)-2(x-2)(x-3)=\frac{3}{2}(x+3)(x-5)$.

(25) $\frac{2}{5}(3x^2-x-5)-\frac{1}{3}(x^2-1)=2(x-2)^2$.

(26) $\cdot 3x^2-\cdot 4x-26=0$.

(27) $\frac{2x^2}{\cdot 1}-\frac{3x}{\cdot 5}=x-\cdot 5$.

(28) $\frac{x}{x+1}-\frac{x+3}{2(x+4)}+\frac{1}{18}=0$.

(29) $x^2-ax+\frac{3a^2}{16}=0$.

(30) $x^2+2ax=3a^2-4ab+b^2$.

(31) $\frac{x+1}{x+4}=\frac{2x-1}{x+6}$.

(32) $3x^4-2x^2-8=\frac{3}{7}(7x^2-2)(x^2-2)$.

(33) $\frac{3x}{2(x+1)}-\frac{5}{8}=\frac{3x^2}{x^2-1}-\frac{23}{4(x-1)}$.

(34) $(x-2)(x-4)-2(x-1)(x-3)=0$.

(35) $\frac{9}{10x}=\cfrac{1}{x+\cfrac{2}{x+3}}$.

(36) $\frac{x}{x+1}+\frac{x}{x+2}+\frac{x}{x+5}+\frac{24}{3x+8}=3$.

(37) $x=1+\cfrac{1}{1+\cfrac{1}{1+x}}$.

3) $(5x-2b)^2-(2a-x)^2=0.$ (39) $x^2+(a+b)x+ab=0.$

)) $(x+2a)(x+2b)=4(x-a)(x-b)+3(a+b)^2.$

---

1) $\dfrac{x+4}{x-4}+\dfrac{x+9}{x-9}=\dfrac{x-4}{x+4}+\dfrac{x-9}{x+9}.$

2) $(x-1)(x-2)(x-3)=0.$

3) $(x+1)(x-2)(x+3)=-6.$

4) $(x+1)(x-2)(x^2+x-2)=0.$

5) $(x^2-3x+2)\,.\,(x^2-x-12)=0.$

6) $x^3-x^2-x+1=0.$ (47) $x(x-a)(x^2-b^2)=0.$

3) $x(x+1)(x+2)=(a-2)(a-1)a.$

9) $x^2-x+a=0.$

Vhat value of a will make the two roots equal to one ther?

0) $x^2-px+q=0.$

f $p=3$, what must be the value of q to make the two ts equal? If $q=9$, what values of p will make them al?

2.—The solution of an equation may be obtained by ipleting the square in two other cases:—

i) When only two powers of x are involved, and one of them is the square of the other.

x. To solve $x^6+7x^3=8$.

Here 6 being double of 3, x^6 is the square of x^3.

$$x^6+7x^3+\left(\frac{7}{2}\right)^2=8+\frac{49}{4}$$
$$=\frac{81}{4},$$
$$\therefore\quad x^3+\frac{7}{2}=\pm\frac{9}{2},$$
$$x^3=1 \text{ or } -8,$$

$\therefore$ taking the cube root, $x=1$ or -2.

ii) Sometimes it will be found by trial that the addition of a number to an equation of the fourth degree will give a complete square on both sides.

Ex. To solve $x^4-2x^3-13x^2+14x+24=0$.

Taking the square root of the left side,

$$\begin{array}{l|l}
 & x^4-2x^3-13x^2+14x+24\ (x^2-x-7 \\
 & x^4 \\
2x^2-x & -2x^3-13x^2 \\
 & -2x^3+\quad x^2 \\
2x^2-2x-7 & -14x^2+14x+24 \\
 & -14x^2+14x+49 \\
 & \qquad\qquad\quad -25
\end{array}$$

we see that, if 25 were added, the square would be completed; we should then have

$$x^4-2x^3-13x^2+14x+49=25,$$

and extracting the square root,

$$x^2-x-7=\pm 5,$$

$$\therefore\quad x^2-x=12 \text{ or } 2,$$

$$x^2-x+\frac{1}{4}=\frac{49}{4} \text{ or } \frac{9}{4},$$

$$x-\frac{1}{2}=\pm\frac{7}{2} \text{ or } \pm\frac{3}{2},$$

$$x=\frac{1\pm 7}{2} \text{ or } \frac{1\pm 3}{2},$$

$$=4 \text{ or } -3 \text{ or } 2 \text{ or } -1.$$

Exercise 37.

Find the *possible* roots of

(1) $x^4-5x^2+4=0$. (2) $37x^2-4x^4=9$.

(3) $8x^6+63x^3=8$. (4) $16x^8-17x^4+1=0$.

(5) $216x^7+19x^4=x$. (6) $32x^{10}+1=33x^5$.

(7) $x^4-4x^3-10x^2+28x-15=0$.

(8) $4x^4-20x^3+23x^2+5x=6$.

(9) $x^5-10x^4+35x^3-50x^2+24x=0$.

(10) $108x^4+51x^2=20x(9x^2-1)+7$.

PROBLEMS PRODUCING QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY.

73.—Problems whose solution gives rise to quadratic equations differ from those which involve simple equations only in this, that, as a quadratic equation has *two* roots, there will be apparently *two* solutions to the problem.

Sometimes both of these will be true solutions; but frequently one only will be a true solution, the other being inconsistent with the conditions implied in the problem.

Ex. (1) Find two consecutive numbers the treble of whose product exceeds four times their sum by 8.

Let x and $x+1$ be the two numbers:

Then $3x(x+1) = 4(2x+1)+8,$

$$3x^2+3x = 8x+4+8,$$

$$3x^2-5a = 12,$$

$$x^2-\frac{5}{3}x = 4,$$

$$x^2-\frac{5}{3}x+\left(\frac{5}{6}\right)^2 = \frac{169}{36},$$

$$x-\frac{5}{6} = \pm\frac{13}{6},$$

$$x=\frac{5\pm 15}{6} = 3 \text{ or } -\frac{4}{3}.$$

The former of these two roots gives 3 and 4 for the numbers required.

The latter is inconsistent with the conditions of the question, *consecutive numbers* being understood to be integers following one another in the common scale, 1, 2, 3, 4, &c.

Ex. (2) Two pieces of cloth, consisting of 6 yards and 14 yards respectively, are bought for £10 12*s.*, and the buyer finds that for £3 he gets one yard more of the latter than of the former: what price per yard does he pay for each piece?

Let x be the price in shillings of the former; then for £3 he buys $\frac{60}{x}$ yards;

∴ of the latter he buys for £3, $\frac{60}{x}+1$ yards, so that the cost of the latter per yard is

$$\frac{60}{\frac{60}{x}+1} = \frac{60x}{60+x} \text{ shillings.}$$

$$\therefore \quad 6x + \frac{60x}{60+x}\times 14 = 212;$$

$$\therefore \quad 360x + 6x^2 + 840x = 1272+212x,$$

$$6x^2 + 988x = 12720,$$

the roots of which equation are 12 and $-176\frac{2}{3}$.

The latter of these roots does not give a solution of the problem; the former gives 12*s.* for the price per yard of the first piece, and $\frac{60x}{60+x}$, *i.e.* 10*s.*, of the second.

Exercise 38.

(1) Find a number such that, when added to its square, the sum may be 20.

(2) Find a number such that, when its double is increased by 3, and diminished by 3, the product of the numbers so obtained may be 135.

(3) The length of a rectangle is 5 feet more than its breadth, and its area is 300 square feet: what are its dimensions?

(4) Divide 100 into two parts such that their product may be 1344.

(5) The area of a square may be doubled by increasing its length by 6 inches, and its breadth by 4 inches: determine its side.

(6) Find three consecutive numbers such that the squares of two of them may be together equal to the square of the third.

(7) There are three lines, the first two of which are each $\frac{4}{7}$ of the third, and the squares described on them are together equal to a square yard: find the lengths of the lines.

(8) Find a number such that, when increased by 12, its square may be increased to $6\frac{1}{4}$ times its former value.

(9) A grass plot 9 yards long and 6 yards broad has a path round it whose area is equal to that of the grass: find its width.

(10) A could do some work alone in 9 hours less than B could do it; and together they could do it in 20 hours: how long would each take?

(11) A person buys for a room as many yards of carpet as he pays pence per yard. For a narrower carpet, the price of which per yard was $\frac{11}{12}$ of the former, but of which he would have required 6 yards more, he would have paid 2*s.* 6*d.* additional. How many yards did he buy, and at what price?

(12) An iron bar weighs 36 lbs. If it had been a foot longer, each foot of it would have weighed half a pound less. Find the length and the weight per foot.

(13) Describe a right-angled triangle, one of whose sides shall be 3 feet more, and another 21 feet less, than the third.

(14) What is the price of oranges per dozen when the number that can be bought for 1*s*. is $6\frac{3}{4}$ times the number of pence which each orange costs?

(15) Divide 35 into two parts, so that the sum of the fractions formed by dividing each part by the other may be $2\frac{1}{12}$.

(16) A number consists of two digits one of which is the square of the other, and, when 54 is added, has its digits inverted. Find it.

(17) The thickness of a rectangular solid is $\frac{2}{3}$ of its breadth, and its length equals its breadth and thickness added together: also the number of cubic yards in its volume added to the whole length in yards of its edges is $\frac{5}{3}$ of the number of square yards in its surface. Determine its dimensions.

(18) A man died in a year A.D. which was $33\frac{1}{4}$ times his age; 13 years before, the year had been the square of his age. How old was he when he died?

(19) A courtyard 40 feet square has its sides paved to a uniform width with stones 2 feet square, of which 204 are used. How wide is the pavement?

(20) Could the same stones be used for paving in a similar manner a courtyard 60 feet by 50 feet, and to what width?

(21) 12 ducks and 8 turkeys cost £5 12*s*.; and 4 more ducks can be bought for 18*s*. than turkeys for 19*s*. Find the price of each.

(22) A courtyard is surrounded by a wall. The length of the courtyard is 4 yards more than six times the height of the wall, its breadth 5 yards more than three times the height, and its area 4 yards less than $\frac{3}{2}$ of that of the wall. Determine the height of the wall.

(23) Two trains travel towards one another from two stations $8\frac{2}{11}$ miles apart. One starts two minutes before

the other, and each travels as many minutes before meeting as the other travels yards per second. Determine their rates per hour.

(24) Divide a line 20 inches long into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.

(25) Land was bought for 1000 guineas; and the buyer, after reserving $1\frac{1}{2}$ acres for himself, sold the remainder at an advance of £7 per acre, receiving for it £1025. How much did he buy, and at what price per acre?

(26) A and B start at the same instant from two points, P and Q, 30 yards apart. A, whose rate is 4 yards per second, runs at right angles to PQ; B, whose rate is 5 yards per second, runs in such a direction that, without changing it, he may just catch A. How long did they run?

(27) A corndealer bought 10 quarters of wheat and 8 quarters of barley for £42, being able to buy for £14 one more quarter of wheat than of barley for £7. What was the price of each per quarter?

(28) Describe a circle which would be doubled by increasing its radius by an inch.

(29) A and B carried between them 70 eggs to market, and by selling them at different prices received each the same sum. If A had carried as many as B, he would have received 1*s.* $10\frac{1}{2}$*d.* for them; while B, if he had carried as many as A, would have received 3*s.* 4*d.* How many had each?

(30) If a carriage-wheel, $16\frac{1}{2}$ feet round, took one second more to revolve, the rate of the carriage per hour would be $1\frac{7}{8}$ miles less. At what rate is the carriage travelling?

CHAPTER XVII.

SIMULTANEOUS QUADRATIC EQUATIONS.

74.—Quadratic equations of two unknown quantities require different methods for their solution, according to the form of the equations.

First Method.—One of the unknowns may sometimes be found in terms of the other, as in (52. *Third Method*), and this value substituted.

This method is applicable when one of the unknowns enters into one of the equations only in its simple power.

Ex. (1) Solve $\begin{cases} 3x^2 - 2xy = 5 & \dots (1) \\ x - y = 2 & \dots (2) \end{cases}$

From (2) $x = y + 2$,

Substitute in (1) $3(y+2)^2 - 2y(y+2) = 5$,

from which $y = -1$ or -7

$\therefore\ x = y + 2 = 1$ or -5.

Second Method.—When the terms which involve x and y in each equation are of the *second* degree, one of the unknowns may be obtained in terms of the other by introducing a new letter, as in the following example:—

Ex. (2) Solve $\begin{cases} x^2 - xy - y^2 = 5 & \dots (1) \\ 2x^2 + 3xy + y^2 = 28 & \dots (2) \end{cases}$

Suppose that $y = vx$, then

(1) becomes $(1 - v - v^2)x^2 = 5 \quad \dots (3)$

(2) becomes $(2 + 3v + v^2)x^2 = 28 \quad \dots (4)$

and dividing (3) by (4)

$$\frac{1 - v - v^2}{2 + 3v + v^2} = \frac{5}{28},$$

$$\therefore\ 28 - 28v - 28v^2 = 10 + 15v + 5v^2,$$

$$-33v^2 - 43v = -18,$$

$$v^2 + \frac{43}{33}v = \frac{18}{33},$$

from which $v = \frac{1}{3}$ or $-\frac{18}{11}$.

Substitute in (3), $\left(1-\frac{1}{3}-\frac{1}{9}\right)x^2=5 \therefore x=\pm\ 3,$

$$\text{or } \left(1+\frac{18}{11}-\frac{324}{121}\right)x^2=5 \therefore x=\pm 11\sqrt{-1},$$

$$\therefore y=\frac{1}{3} \text{ of } \pm 3=\pm 1,$$

$$\text{or } -\frac{18}{11} \text{ of } \pm 11\sqrt{-1}=\mp 18\sqrt{-1},$$

$\therefore x=\pm 3,\ y\pm 1$ are the possible roots.

The above method is also applicable when the terms involving $x+y$ in one equation are of the first and in the other of the second degree.

Ex. (1) may be solved in this way:

(1) becomes $(3-2v)x^2=5,$

(2) *squared* becomes $(1-2v+v^2)x^2=4,$

$$\therefore \frac{3-2v}{1-2v+v^2}=\frac{5}{4},$$

from which $\qquad v=-1 \text{ or } \frac{7}{5}.$

N.B.—Care must be taken in this case to substitute these values of v in the *simple* equation (2),

$$(1-v)x=2,$$

from which $\qquad x=1 \text{ or } -5,$

$$\therefore y=-1 \text{ or } -7.$$

Third Method.—Equations from which x^2+y^2 and xy can be obtained may be also conveniently solved as follows:—

Ex. (3) Solve $x^2+y^2=13,\ xy=6$

Add twice the second equation to the first, then

$$x^2+2xy+y^2=25 \quad . \quad . \quad . \quad . \quad (3)$$

Take the square root,

$$x+y=\pm 5 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Subtract twice the second equation from the first,

$$x^2-2xy+y^2=1 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the square root of which gives

$$x-y=\pm 1 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(6) added to (4) will give

$$2x=\pm 6 \text{ or } \pm 4, \therefore x=\pm 3 \text{ or } \pm 2;$$

(6) subtracted from (4) will give

$$2y=\pm 4 \text{ or } \pm 6, \therefore y=\pm 2 \text{ or } \pm 3.$$

N.B.—Care must be taken in the last addition and subtraction to use the double signs in (4) and (6) in the *same order.*

Exercise 39.

Solve—

(1) $x+y = 5,$
$x^2-3y^2=-3.$

(2) $x-3y=1,$
$xy+y^2=5.$

(3) $x+y = 1,$
$x^2+y^2=25.$

(4) $x-y = 7,$
$x^2+xy+y^2=13.$

(5) $x^2+xy=35,$
$xy-y^2= 6.$

(6) $xy =12,$
$x-2y= 5.$

(7) $xy = 7,$
$x^2+y^2=50.$

(8) $2x-5y =9,$
$x^2-xy+y^2=7.$

(9) $x-y =9,$
$xy+8=0.$

(10) $x-y =1,$
$x^2+y^2=8\frac{1}{2}.$

(11) $4y =5x+1,$
$2xy=33-x^2.$

(12) $x^2+4xy=3,$
$4xy+y^2 =2\frac{1}{4}.$

(13) $5x-7y =0,$
$5x^2-\frac{13}{4}xy+7y^2=4.$

(14) $x^2-xy+y^2=48,$
$x-y = 8.$

(15) $x-y=1,$
$\frac{x}{y}+\frac{y}{x}=2\frac{1}{6}.$

(16) $x^2+3xy+y^2 =-1,$
$3x^2+xy+3y^2= 13.$

(17) $x^2-2xy+3y^2=1\frac{2}{9},$
$x^2+xy-y^2 = \frac{1}{9}.$

(18) $x+y=a,$
$4xy =a^2-4b^2.$

(19) $x^2-xy=a^2+b^2,$
$xy-y^2=2ab.$

(20) $x^2-y^2=4ab,$
$xy =a^2-b^2.$

(21) $\frac{1}{x} + \frac{1}{y} =5,$
$\frac{1}{x+1}+\frac{1}{y+1}=1\frac{5}{12}.$

(22) $\frac{x+y}{x-y}+\frac{x-y}{x+y}=2\frac{9}{40},$
$6x=20y+9.$

(23) $x^3-y^3=9,$
$x-y =3.$

(24) $x^2-y^2 =0,$
$3x^2-4xy+5y^2=9.$

(25) $xy = 0,$
$x^2+y^2=16.$

(26) $x^3-y^3=a^3,$
$x-y =a.$

(27) $\frac{3}{5}(x-1)-\frac{2}{3}(x+1)\ (y-1)+11=0,$
$\frac{1}{3}(y+2)=\frac{1}{4}(x+2).$

(28) $ax+by =0,$
$(x+y)^2+4=\frac{(a-b)\ (a^3-b^3)}{a^2b^2}-xy.$

(29) $10x^2+15xy=3ab-2a^2,$
$10y^2+15xy=3ab-2b^2.$

(30) Find y and z in terms of x from

$$\left.\begin{aligned}5x^2-y^2-z^2&=0\\x+y+z&=0\end{aligned}\right\}$$

Also from

$$\left.\begin{aligned}xy+yz+zx&=0\\xyz&=a^3\end{aligned}\right\}$$

PROBLEMS PRODUCING QUADRATIC EQUATIONS OF TWO UNKNOWN QUANTITIES.

Exercise 40.

(1) If the length and breadth of a rectangle were each increased by 1, the area would be 48; if they were each diminished by 1, the area would be 24; determine the length and breadth.

(2) The hypothenuse of a right-angled triangle is 20, and its area 96; find its sides.

(3) The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two is $1\frac{5}{12}$; if the numerators were interchanged, the sum of the fractions would be $1\frac{1}{2}$: what are the fractions?

(4) The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more; find the circumference of each.

(5) The sum of two numbers which are formed by the same two digits in reverse orders is $\frac{45}{18}$ of their difference; and the difference of the squares of the numbers is 3960; what are they?

(6) Two boys run in opposite directions round a rectangular field the area of which is an acre; they start from one corner and meet 13 yards from the opposite corner, and the rate of one is $\frac{5}{6}$ of that of the other. Determine the dimensions of the field.

(7) On three successive days the rate of the stream in a river is 2 miles, $1\frac{1}{2}$ mile, 1 mile an hour; what distance

up the stream and back does a person row who takes on the last two days 25′ and 32′ less than on the first?

(8) A number is divided into two parts such that the sum of the first and the square of the second is twice the sum of the second and the square of the first; and the sum of the number and the first part is 4 more than twice the second. Find the number.

(9) A man and a boy begin together to clear a piece of ground. After a time the man leaves the boy to finish, who works in consequence for a fourth as long again as he would have done if the man had not left. If they had worked together to the end, the man would have cleared 10 yards more than he did, and would have done as much as the boy could do in 32 hours; while the boy would have done as much as the man could do in 8 hours. How large was the piece?

(10) A, in running a race with B to a post and back, met him 10 yards from the post. To make it a dead heat, B must have increased his rate from this point $41\frac{3}{7}$ yards per minute; and if without changing his pace he had turned back on meeting A, he would have come in 4″ after him. How far was it to the post?

CHAPTER XVIII.

SIMPLE INDETERMINATE EQUATIONS.

75.—If a single equation be given which involves two unknown quantities, and no other condition be assigned, the number of its solutions (51) will be unlimited; for any value may be assumed for one unknown and a corresponding value then found for the other.

Such an equation is said to be *Indeterminate.*

An indeterminate equation differs from an *identity* (20) in this, that whereas in an identity, *e.g.*

$$\frac{x+1}{2}+\frac{2y-5}{3}=3-\frac{5+4y-3x}{6},$$

any values may be assigned to x and y independently of one another, in an indeterminate equation the values of x and y are dependent one on the other; and so while they are unlimited in number, they are still confined to a particular range.

This range is frequently still further limited by requiring the roots to satisfy some given condition, as, for instance, that they should be *positive integers:* the manner of introducing such conditions will be best seen by examples.

Ex. (1) Solve $3x+4y=22$ in positive integers.

Transposing, $\quad 3x=22-4y,$

$$\therefore\ x=7-y+\frac{1-y}{3},$$

the quotient being written as a mixed number.

$$\therefore\ x+y-7=\frac{1-y}{3}.$$

Now since the values of x and y are to be integral, $x+y-7$ will be integral, and therefore $\frac{1-y}{3}$, though fractional in *form*, will be integral.

Suppose $\frac{1-y}{3}$ to be $=t$, an integer,

then $\qquad 1-y=3t,$

and $\qquad y=1-3t \ldots\ldots\ldots (\alpha).$

$$\therefore\ x=7-y+t,$$
$$=7-1+3t+t,$$
$$=6+4t \ldots\ldots\ldots (\beta).$$

It is clear from (α) and (β) that if t have any integral value (1, 2, 3, &c.) positive or negative, x and y will both be *integral:* and such values must be chosen as will also make x and y *positive.*

(α) shows that so far as regards y, t may have any negative value, or 0, but cannot have a positive value assigned.

(β) shows that t may not have any negative value greater than 1.

$$\therefore\ t \text{ may } = -1 \text{ or } 0, \text{ and then}$$
$$x=2,\ y=4,$$

or $\qquad x=6,\ y=1.$

Ex. (2) Solve $5x-7y=13$ in positive integers.

Proceeding as above, we find that

$$\frac{2y+3}{5} \text{ must be integral.}$$

Now if this were put $= t$, as before, the value of y obtained in terms of t as in (α) would be *fractional*, and therefore useless. To avoid this difficulty, multiply the above expression by 3, and we shall have

$$\frac{6y+9}{5} \text{ i.e. } y+1+\frac{y+4}{5} \text{ integral;}$$

$$\therefore\ \frac{y+4}{5} \text{ is integral} = t \text{ (suppose);}$$

$$\therefore\ y=5t-4,$$

and $\qquad x=7t-3.$

Here t may have any positive value,

and $\qquad x=4,\ 11,\ 18,\ 25,\ \ldots.$

$$y=1,\ 6,\ 11,\ 16,\ \ldots.$$

The multiplier to be employed as above must be such that the resulting coefficient of y, when divided by the denominator, shall have a remainder 1; such a multiplier may always be found if the coefficient and the denominator are prime to one another.

It will be noticed that the values of x and y are limited when, as in Ex. (1), t has *different* signs in (α) and (β).

When t has the same sign in both (*i.e.* when x and y in the given equation are connected by $-$), the number of their values is unlimited.

The necessity for a multiplier may be sometimes obviated in one of these ways.

Ex. (3) Solve $7x+4y=29$ in positive integers.

$$4y=29-7x.$$

Divide by 4, and *notice that* 7 *is* 1 *less than a multiple of* 4; then

$$y=7-2x+\frac{1+x}{4},$$

&c.

Ex. (4) Solve $5x-13y=8$ in positive integers.

$$x=2y+1+\frac{3y+3}{5}.$$

Here the numerator of the fraction contains a factor 3: dividing by it, we have

$\frac{y+1}{5}$ an integer, $=t$ (suppose),

&c.

When one pair of roots can be determined by inspection, the rest may also be found as in the following example.

Ex. (5) Solve $3x+5y=52$ in positive integers.

Here $x=4$, $y=8$ are a pair of roots;
for $3\cdot4+5\cdot8=52$.

Subtracting the last line from the given equation,

$$3(x-4)+5(y-8)=0.$$

$$\therefore \frac{x-4}{5}=\frac{8-y}{3}=t \text{ (suppose)}.$$

$$\therefore x=5t+4 \quad \ldots\ldots (\alpha)$$
$$y=8-3t \quad \ldots\ldots (\beta)$$

t may $=0, 1, 2.$

$$\therefore x=4, 9, 14,$$
$$y=8, 5, 2.$$

Since all the values of x and y are to be integral, they will differ by integers; (α) and (β) therefore show that the only admissible values of t will be integral.

Ex. (6) Solve $18x-5y=70$, so that y may be a multiple of x, and both positive.

Let $\qquad y=mx$, then

$$(18-5m)x=70$$

$$x=\frac{70}{18-5m}.$$

If $m=2$, $\qquad x=8\frac{3}{4},\ y=17\frac{1}{2}$;

If $m=3$, $\qquad x=23\frac{1}{3},\ y=70.$

If $m=4$, $\qquad x$ is negative.

When a single equation involving three unknown quantities is given, successive values may be assigned to one of them, and corresponding values then found as above for the other two.

Exercise 41.

Solve in positive integers.

(1) $2x+11y=49$. (2) $7x+3y=40$.

(3) $5x+7y=53$. (4) $x+10y=29$.

(5) $3x+8y=61$. (6) $8x+5y=97$.

(7) $16x+7y=110$. (8) $7x+10y=206$.

(9) $5x-14y=11$. (10) $12x-7y=1$.

Find the least positive integers which satisfy the equations:—

(11) $5x-17y=23$. (12) $23y-13x=3$.

(13) In how many ways may 100 be divided into two parts, one of which shall be a multiple of 7 and the other of 9?

(14) Some men earning 2*s*. 6*d*. and some women earning 1*s*. 9*d*. receive altogether for their daily wages £2 4*s*. 9*d*.: how many are there of each?

(15) 11 lbs. of tea and 5 lbs. of coffee are bought for £2 4*s*.; tea being dearer than coffee, find the price per lb. of each.

(16) Solve $5x+6y=30$, so that x may be a multiple of y, and both positive.

(17) Solve $12x+8y=20$, so that $x+y$ may be a positive integer.

(18) Solve $8x+12y=23$, so that x and y may be positive, and their sum an integer.

(19) Solve $5x+9y=41$, so that x and y may be positive, and $2x+3y$ an integer.

(20) Solve $\left.\begin{aligned} 2x+3y+4z=27 \\ 3x+5y+7z=44 \end{aligned}\right\}$ in positive integers.

(21) Solve $3x+5y+10z=31$ in positive integers.

(22) Divide 70 into three parts which shall be multiples respectively of 6, 7, 8.

(23) 200 is to be divided into parts which shall be respectively divisible by 5, 7, 11; and the sum of their quotients to be 20. Show that it can be done only in one way, and find it.

(24) A number consisting of three digits of which the middle one is 4, has its digits inverted by adding 792: determine it.

(25) If $7x+5y=70$ be solved in positive roots, show that any pair of corresponding roots must have its sum greater than 10 and less than 14.

(26) A number of lengths, 3 feet, 5 feet, and 8 feet, are cut; how may 48 of them be taken so as to measure 175 feet altogether?

(27) A field containing an exact number of acres less than 10 is divided into 8 allotments of one size and 7 of four times that size, and has as well a road containing 1300 square yards passing through it. How large are the allotments?

(28) A square and regular nonagon are to be made, such that the sum of their sides (one of each) may be 28 inches, and the sum of their perimeters an exact number of feet. How long must their sides be?

(29) Two wheels are to be made, the circumference of one of which is to be a multiple of the other; what circumferences may be taken so that when the first has gone round 3 times and the other 5, the difference in the length of rope coiled on them may be 17 feet?

(30) The side of a room $27\frac{3}{4}$ feet long is to be panelled with boards, some of which are 8 inches and the rest 5 inches wide. How many of each must be taken (i) in order that there may be more narrow than broad, but the difference the least possible; (ii) that there may be the least possible number on the whole used; (iii) in order that the number used may be the largest possible?

CHAPTER XIX.

RATIO, PROPORTION, AND VARIATION.

76.—The relation which one quantity has to another, in reference to the *number of times* which the former contains the latter, is called the *Ratio* of the former to the latter; and is represented by the fraction which the former is of the latter.

The ratio of a to b is therefore *measured by* the fraction $\frac{a}{b}$; and this fraction is commonly said to *be* the ratio.

The ratio of a to b is also expressed by $a : b$.

It will be observed that a quantity cannot have a ratio to another unless it is *of the same kind* with it.

The first term of a ratio is called the *antecedent*, and the second the *consequent*. If the antecedent is equal to the consequent, the ratio is said to be one *of equality;* if the antecedent is greater than the consequent, it is said to be a ratio *of greater inequality;* if less, *of less inequality.*

The ratio $b : a$ is called the *inverse* of $a : b$.

77.—As propositions respecting ratios will be proved by means of the fractions which represent them, it may be well to recapitulate the leading arithmetical principles respecting fractions, and to give their proofs in an algebraical form.

(i) A fraction will not be altered if its numerator and denominator are both multiplied by the same integer.

For $\frac{a}{b}$ implies that the unit has been divided into b equal parts, and a of them taken.

Now if each of the b parts be subdivided into m equal bits, there will be clearly mb of these in the whole, and ma of them in the above portion.

The portion may therefore be also represented by $\frac{ma}{mb}$.

$$\therefore \frac{a}{b} = \frac{ma}{mb}.$$

(ii) Two fractions having the same denominator will be *added* or *subtracted* by adding or subtracting their numerators, and retaining the denominator.

For $\frac{a}{b}$ indicates that the unit is divided into b parts and a of them taken;

$\frac{c}{b}$ indicates c of the same parts:

$\therefore \frac{a}{b}+\frac{c}{b}$ consists of a parts and c parts of that size,

and therefore $= \frac{a+c}{b}$.

Similarly $$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}.$$

It is clear also that $\frac{a}{b} \gtrless \frac{c}{b}$ according as $a \gtrless c$.

(iii) A fraction will be *multiplied* by an integer if its numerator be multiplied by it.

For $\frac{a}{b} \times m$ means $\frac{a}{b}+\frac{a}{b}+\frac{a}{b} \ldots m$ times

$$= \frac{a+a+a \ldots m \text{ times}}{b},$$

$$= \frac{ma}{b}.$$

(iv) A fraction will be *divided* by an integer if its denominator be multiplied by it.

For since $$\frac{a}{mb} \times m = \frac{ma}{mb} = \frac{a}{b},$$

$$\therefore \frac{a}{b} \div m = \frac{a}{mb}.$$

(v) Fractions having different denominators may be compared, added, or subtracted by expressing both with a common denominator; any common multiple of the denominators being taken for the common denominator.

For two fractions $\frac{a}{b}$, $\frac{c}{d}$ may be expressed as $\frac{ad}{bd}$, $\frac{bc}{bd}$; and these having the same denominator may be compared, added, or subtracted by (ii).

78.—PROP. *A ratio will not be altered if its terms are both multiplied by the same integer.*

For the ratio $a : b$ is represented by $\frac{a}{b}$,

and $ma : mb$ is represented by $\frac{ma}{mb}$,

and by (77. i) $\frac{ma}{mb} = \frac{a}{b}$.

79.—PROP. *A ratio will be altered if its terms are multiplied by different integers: increased or diminished according as the multiplier of the antecedent is greater or less than that of the consequent.*

For if $ma : nb$ be the new ratio,

$$\frac{ma}{nb} \text{ is } \gtrless \frac{a}{b} \left(\text{i. e. } \frac{na}{nb}\right),$$

$$\text{according as } ma \text{ is } \gtrless na,$$

$$\text{or } m \gtrless n.$$

80.—PROP. *A ratio of greater inequality will be diminished, and a ratio of less inequality increased, by adding the same number to both its terms.*

For $a+x : b+x$ will be $\gtrless a : b$,

$$\text{according as } \frac{a+x}{b+x} \gtrless \frac{a}{b},$$

$$\text{as } ab+bx \gtrless ab+ax,$$

$$\text{as } bx \gtrless ax,$$

$$\text{as } b \gtrless a.$$

Similarly *a ratio of greater inequality will be increased, and of less inequality diminished, by subtracting the same number from both its terms.*

81.—A ratio is said to be *compounded* of two others when its terms are respectively the products of their corresponding terms.

The ratio compounded of $a : b$ and $c : d$ is $ac : bd$.

A ratio, $a : b$, compounded with itself produces its

duplicate, $a^2 : b^2$; compounded with itself again, its triplicate, $a^3 : b^3$; &c.

N.B.—The *duplicate, triplicate,* &c. of a ratio are sometimes called its *double, triple,* &c.

Prop. *If a series of magnitudes* a, b, c, d, *are taken, the ratio* $a : d$ *is equal to that compounded of* $a : b$, $b : c$, $c : d$.

For this compound ratio is $abc : bcd$,

$$\text{which} = a : d.$$

82.—An equality of two ratios is called a *proportion*, and the terms involved are said to be *proportionals*.

Thus if $a : b = c : d$, or (as it is more commonly written) $a : b :: c : d$,

$$a, b, c, d \text{ are } proportionals.$$

The first and last terms of a proportion are called the *extremes*, and the two middle terms the *means*.

Prop. *In a proportion between numbers, the product of the extremes is equal to the product of the means.*

For if $\qquad a : b :: c : d$,

$$\frac{a}{b} = \frac{c}{d},$$

$$\therefore \text{ multiplying by } bd, \; ad = bc.$$

Conversely, if $ad = bc$, we obtain by dividing both sides by bd,

$$\frac{ad}{bd} = \frac{bc}{bd},$$

$$\text{or, } \frac{a}{b} = \frac{c}{d}.$$

$$\therefore a : b :: c : d.$$

The equation $ad = bc$ gives $a = \dfrac{bc}{d}$, $b = \dfrac{ad}{c}$, &c.: so that three terms of a proportion being given the other one may be found: an extreme may be found by dividing the product of the means by the other extreme, a mean by dividing the product of the extremes by the other mean.

83.—If $a : b :: c : d$, several other proportions may be formed:—

Inversely; i.e. $b : a :: d : c$ . . . . (i)

$$\text{For } \frac{a}{b} = \frac{c}{d}, \quad \therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d},$$

$$\text{i.e. } \frac{b}{a} = \frac{d}{c},$$

$$\therefore b : a :: d : c.$$

Componendo; i.e. $a+b:b::c+d:d$ . . . (ii)

$$\text{For } \frac{a}{b}=\frac{c}{d},$$

$$\therefore \frac{a}{b}+1=\frac{c}{d}+1,$$

$$\text{or } \frac{a+b}{b}=\frac{c+d}{d},$$

$$\therefore a+b:b::c+d:d.$$

Dividendo; i.e. $a-b:b::c-d:d$ . . . (iii)

$$\text{For } \frac{a}{b}-1=\frac{c}{d}-1, \text{ \&c.}$$

Also $a+b:a-b::c+d:c-d$ . . . . . (iv)

$$\text{For, as above, } \frac{a+b}{b}=\frac{c+d}{d},$$

$$\text{and } \frac{a-b}{b}=\frac{c-d}{d},$$

$\therefore$ dividing the former line by the latter,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d},$$

$$\therefore a+b:a-b::c+d:c-d.$$

When the four quantities are all of the same kind, they will be proportionals also when taken *alternately*; i.e.

$a:c::b:d$ . . . . (v)

$$\text{For } \frac{a}{b} = \frac{c}{d},$$

$$\therefore \frac{a}{b}\times\frac{b}{c} = \frac{c}{d}\times\frac{b}{c},$$

$$\text{or } \frac{a}{c} = \frac{b}{d},$$

$$\therefore a:c::b:d.$$

and from this a series of proportions may be obtained in the same manner as before.

84.—PROP. If $a:b::c:d::e:f$,
then $a:b::a+c+e:b+d+f$.

We shall employ here a method of proof which is often useful for more general propositions.

$$\text{Let } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r,$$

then $a = br$, $c = dr$, $e = fr$;

$\therefore\ a+c+e = (b+d+f)r.$

$$\therefore\ \frac{a+c+e}{b+d+f} = r = \frac{a}{b}.$$

$\therefore\ a : b :: a+c+e : b+d+f.$

Similarly $a : b :: ma+nc+pe : mb+nd+pf.$

85.—PROP. If $a, b, c, d, \ldots\ldots$ are *continued* portionals, *i.e.* if $a : b :: b : c :: c : d ::$ &c.

then $a : c :: a^2 : b^2$, $a : d :: a^3 : b^3$, &c.

$$\text{For } \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \text{\&c.}$$

$$\therefore\ \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b},$$

$$\text{or } \frac{a}{c} = \frac{a^2}{b^2}\ \therefore\ a : c :: a^2 : b^2.$$

$$\text{So } \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b},$$

$$\text{or } \frac{a}{d} = \frac{a^3}{b^3}\ \therefore\ a : d :: a^3 : b^3, \text{\&c.}$$

If three quantities a, b, c, are proportional, so $a : b :: b : c$, b is said to be a *mean* proportional bet\ a and c; and c a third proportional to a and b.

86.—The propositions proved above should be membered also in their fractional form; and (espec in this form) are often useful for the simplificatio equations.

$$\text{Ex. Solve } \frac{3x^2+2x+6}{3x^2-2x-6} = \frac{3x^2+2x-20}{3x^2-2x-16}.$$

Instead of multiplying out at once, we may apply (83 from which

$$\frac{6x^2}{4x+12} = \frac{6x^2-36}{4x-4}.$$

$$\therefore\ \frac{x^2}{x+3} = \frac{x^2-6}{x-1},$$

$$x^3-x^2 = x^3+3x^2-6x-18,$$

$$4x^2-6x = 18,$$

from which $x = 3$ or $-1\frac{1}{2}$.

Equations which are given in the form of a proportion may be solved either by being expressed in a fractional form or by equating the products of the extremes and means (82).

87.—PROP. If $a : b :: c : d$, and a is the greatest term, then $a+d$ will be $> b+c$.

For $\frac{a}{b} = \frac{c}{d}$ $\therefore$ a being $> c$, b will be $> d$.

Also $\frac{a-b}{b} = \frac{c-d}{d}$, and b being $> d$, $a-b$ will be $< c-d$,

$\therefore$ adding $b+d$, $a+d > b+c$.

Exercise 42.

(1) Write down the ratio compounded of $3 : 5$ and $8 : 7$; which of these ratios is increased and which diminished by the composition?

(2) Compound the duplicate ratio of $4 : 15$ with the triplicate of $5 : 2$.

(3) Show that a duplicate ratio is greater or less than its simple ratio, according as it is a ratio of greater or less inequality.

(4) Find two numbers in the ratio $3 : 5$ such that when 5 is added to each, they may be in the ratio $2 : 3$.

(Let $3x$, $5x$ be the numbers.)

(5) A, B, C had sums proportional to 4, 5, 12; and when C had given A and B each 5*s.* 6*d.*, the sum which he had left bore to the sum which they had together the ratio $2 : 7$. What sum had each at first?

(6) If $a : b :: c : d$, prove that

(i) $ma : nb :: mc : nd$.
(ii) $3a+b : b :: 3c+d : d$.
(iii) $a+2b : b :: c+2d : d$.
(iv) $a^2 : b^2 :: c^2 : d^2$.
(v) $a : a+b :: c : c+d$.
(vi) $a : a-b :: c : c-d$.
(vii) $ma+nb : ma-nb :: mc+nd : mc-nd$.
(viii) $a^m : b^m :: c^m : d^m$.

(7) If $a:b::c:d$, prove (by the method of 84) that

(i) $2a+3b : 3a-4b :: 2c+3d : 3c-4d$.

(ii) $a^2+ab : b^2-ab :: c^2+cd : d^2-cd$.

(iii) $ma^2+nc^2 : mb^2+nd^2 :: a^2 : b^2$.

(iv) $ma^2+nab+pb^2 : b^2 :: mc^2+ncd+pd^2 : d^2$.

(8) If $\frac{ax+cy}{ay+cx}=\frac{bx-cy}{by-cx}$, show by (84) that each of these fractions $=\frac{x}{y}$.

(9) If $a:b::b:c$, prove that

(i) $ac=b^2$.

(ii) $a+b : b+c :: a : b$.

(iii) $a^2+ab : b^2+bc :: a : c$.

(iv) $a : b :: (a+c)^2 : (b+c)^2$.

(10) If a, b, c are proportionals, and a the greatest, then $a+c > 2b$.

(11) If $\frac{x-y}{l}=\frac{y-z}{m}=\frac{z-x}{n}$ and x, y, z unequal, then $l+m+n=0$.

(12) Find x when $x+5 : 2x-3 :: 5x+1 : 3x-3$.

(13) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+c}{b+d}$ or $\frac{a-c}{b-d}$ will equal either of these: if $\frac{a}{b}>\frac{c}{d}$, then $\frac{a+c}{b+d}$ is $<\frac{a}{b}$ but $>\frac{c}{d}$.

(14) Solve $\frac{x^2+x+1}{x^2-x-1}=\frac{x^2-x+2}{x^2+x-2}$; and

$$\frac{x^2-4x+2}{x^2-2x-1}=\frac{x^2-4x}{x^2-2x-2}.$$

(15) If $\frac{a+b+c+d}{a-b+c-d}=\frac{a+b-c-d}{a-b-c+d}$, then $a:b::c:d$.

(16) A line is divided into two parts in the ratio $2:3$, and into two parts in the ratio $3:4$; and the distance between the points of section is 2. Find the length of the line.

(17) A railway consists of two sections. The annual expenditure on one has increased this year 5 per cent., and on the other 4 per cent., producing on the whole an increase of $4\cdot3$ per cent. Compare the amounts expended on the two sections (i) last year, (ii) this year.

(18) A certain number of persons, consisting of men and women earning respectively fixed rates of wages, are required for some work. The numbers of men and women are equal, but if the number of women were made half as much again as the number of men, the amount of wages paid would be a thirtieth less than at present. Compare the wages of a man and woman.

(19) If a, b, c, d are proportionals and unequal, show that no number x can be found such that $a+x$, $b+x$, $c+x$, $d+x$ shall be proportionals.

(20) In a factory, the number of working hours per day having been reduced in the ratio $11:9$, and the number of working days per week in the ratio $12:11$, it is found that a workman who is paid proportionately to the work done, has increased his weekly wages 2 per cent. In what ratio has the value of his work per hour increased?

VARIATION.

88.—Quantities may be so connected that when one has its value changed, the other will in consequence have its value changed also.

> Thus the distance travelled at a certain rate will be increased if the time is increased; the time required for completing a certain quantity of work will be diminished if the number of workmen is increased.

It will be noticed that the word "quantity" is here used in a sense rather different from that in which it has been before employed. Hitherto, when a, b, x inches, y men, &c. have occurred, the letters (or quantities) have been considered to have *particular values* throughout any one proposition or operation in which they have occurred: here, however, the word is used in a *general* sense, as *distance*, *area*, &c., to which particular values may be assigned.

To indicate quantities understood in this sense it is convenient to use capital letters, A, B, &c., the letters being understood to indicate *numerical* values of the quantities. When two such letters are used in an expression, they indicate *corresponding* values of the two quantities.

89.—When two quantities are so connected that if one be changed in any ratio the other will also become changed in the same ratio, the latter is said to *vary*, or *vary directly*, as the former.

For instance, the area of a rectangle described on a given base will vary directly as the altitude; for if the altitude is changed, the area will become changed in the same ratio.

Variation is expressed by the symbol $\propto$.

Thus if H and A represent corresponding values of the altitude and area of the rectangle, $A \propto H$.

Hence if $A \propto B$, two values of A have to one another the same ratio as the two corresponding values of B.

PROP. *If* A $\propto$ B, A *will equal* mB, *where* m *is constant for all values of* A *and* B.

For if $\begin{matrix} a, a', a'', \\ b, b', b'', \end{matrix}$ . . . be corresponding values of A and B,

we have from above $\dfrac{a}{a'} = \dfrac{b}{b'}$; $\dfrac{a}{a''} = \dfrac{b}{b''}$; &c.

$$\therefore \frac{a}{b} = \frac{a'}{b'} = \frac{a''}{b''} = \text{\&c.}$$

i.e. the ratio $\dfrac{A}{B}$ is constant, and may be represented by a constant m; $\therefore A = mB$.

The constant m may be determined whenever a pair of corresponding values of A and B is known.

90.—When two quantities are so connected that if one be changed in any ratio, the other will become changed in the inverse ratio, the latter is said to vary *inversely* as the former.

For instance, the time required for completing a certain amount of work varies inversely as the number of workmen employed; for if this be doubled, trebled, halved, or changed in any ratio, the time required will be changed into the half, third, double, or in the inverse ratio.

Hence, if A varies inversely as B, two values of A have to one another the inverse of the ratio which the corresponding values of B have.

PROP. *If* A *vary inversely as* B, A·B *will equal* m, *where* m *is constant for all values of* A *and* B.

For if $\begin{matrix} a, a', a'', \ldots \\ b, b', b'', \ldots \end{matrix}$ be corresponding values of A and B,

$$\frac{a}{a'} = \frac{b'}{b}; \quad \frac{a}{a''} = \frac{b''}{b}; \quad \&c.$$

$$\therefore ab = a'b' = a''b'' = \&c.$$

i. e. AB is constant, and may be represented by m.

As before, m may be determined if one pair of corresponding values of A and B is known.

Instead of $AB = m$, the equation may be written $A = \frac{m}{B}$; and A may be said to vary as the reciprocal of B, or $A \propto \frac{1}{B}$.

91.—Questions involving the principle of Variation may be most easily treated by means of the above equations between the varying quantities.

Ex. (1) If $A \propto B$ and when $B = 2$, $A = 3$, find the value of A when $B = 5$.

Let $A = mB$, then 3 and 2 being corresponding values of A and B,

$$3 = m \times 2 \qquad \therefore \quad m = \frac{3}{2},$$

$$\therefore \qquad A = \frac{3}{2}B,$$

and when $B = 5$, $A = \frac{3}{2}$ of $5 = 7\frac{1}{2}$.

92.—PROP. *If* A$\propto$B, *and* B$\propto$C, *then* A$\propto$C.

For $\quad A = mB$ where m is constant,
and $\quad B = nC$ where n is constant.
$\therefore \quad A = mnC$.
$\therefore \quad mn$ being constant, $A \propto C$.

Ex. (2) The weight of a sphere of given material varies as its volume, and its volume as the cube of its radius. If a sphere of 2 in. radius weigh 5 lbs., find the weight of a sphere of 3 in. radius.

If W represent the weight,
V . . . the volume,
R . . . the radius,
$W \propto V$ and $V \propto R^3$,
$\therefore$ $W \propto R^3$

Let $W = mR^3$,

then 5 and 2 being corresponding values of W and R,

$$5 = 8m, \quad \therefore m = \frac{5}{8}.$$

$$\therefore \quad W = \frac{5}{8}R^3,$$

$$\therefore \quad \text{when } R = 3, \ W = \frac{5}{8} \text{ of } 27 = 16\frac{7}{8}.$$

93.—A quantity may vary jointly as *two* others, *i.e.* it may be so dependent on them that when either of them is changed in any ratio, the other remaining fixed, it will be changed in the same ratio.

Prop. *If* $A \propto B$ *when* C *is unchanged, and* $A \propto C$ *when* B *is unchanged, then when both* B *and* C *change,* $A \propto BC$.

Let a, b, c be simultaneous values of A, B, C.

And let any change of b into b' (c remaining unchanged) produce in a a change into α,

then $$\frac{a}{\alpha} = \frac{b}{b'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now (B remaining b') let c be changed into any value c'; it will produce in α a change into a', such that

$$\frac{\alpha}{a'} = \frac{c}{c'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then the changes in *both* b and c have changed a into a'; and multiplying (1) and (2),

$$\frac{a}{a'} = \frac{bc}{b'c'}$$

Hence the values of A have the same ratio as the corresponding values of the product BC,

$$\therefore \quad A \propto BC.$$

94.—Prove in the same manner the following propositions:—

(i) *If* A *vary directly as* B *when* C *is unchanged, and inversely as* C *when* B *is unchanged, then when both* B *and* C *change,* $A \propto \frac{B}{C}$.

(ii) *If* A *vary inversely as* B *when* C *is unchanged, and inversely as* C *when* B *is unchanged, then when both* B *and* C *change,* A *varies inversely as* BC.

(iii) *If there are any number of quantities* B, C, D, *&c., and* A *varies as each when the rest are unchanged, then when they all change,* $A \propto BCD \dots$

Exercise 43.

(1) If $A \propto B$, and $A = 4$ when $B = 5$, express A in terms of B; and find A when $B = 12$.

(2) If $A \propto B$, and when $B = \frac{1}{2}$, $A = \frac{1}{3}$, find A when $B = \frac{1}{3}$.

(3) If $A \propto B + c$ where c is constant, and $A = 2$ when $B = 1$, and $A = 5$ when $B = 2$, find A when $B = 3$.

(4) If x varies jointly as y and z, and 3, 4, 5, are simultaneous values of x, y, z, find x when $y = z = 10$.

(5) The velocity acquired by a stone falling from rest varies as the time of falling, and the distance fallen varies as the square of the time; if it be found that in 3″ a stone has fallen 145 ft. and acquired a velocity of $96\frac{2}{3}$ ft. per second, find the velocity and the distance at the end of 5″.

(6) If a heavier weight draw up a lighter one by means of a string passing over a fixed wheel, the space described in a given time will vary directly as the difference between the weights and inversely as their sum. If 9 oz. draw 7 oz. through 8 ft. in 2″, how high will 12 oz. draw 9 oz. in the same time?

(7) The space will vary also as the square of the time; find the space if the time in the latter case be 3″.

(8) The expenses of a hospital are partly constant and partly vary as the number of patients; if for 120 patients the expenses are £400, and for 140 patients £430; find the expenses for 200 patients.

(9) Equal volumes of iron and copper are found to

weigh 77 and 89 oz. respectively; find the weight of 10½ ft. of copper rod when 9 in. of iron rod of the same thickness weigh 31·9 oz.

(10) The volume of equal lengths varying as the square of the thickness, find the weight as above, supposing that the thickness of the iron rod were 1 in., and of the copper ¼ in.

(11) The square of a planet's time of revolution varies as the cube of its distance from the sun; the distances of the earth and Mercury from the sun being 91 and 35 millions of miles, find (in years) the time of Mercury's revolution.

(12) The volume of a sphere varies as the cube of its diameter; show that a sphere of 6 in. diameter will be equal in volume to three spheres whose diameters are 3, 4, 5 in.

(13) x is equal to the sum of two quantities, one of which varies inversely as y, and the other inversely as y^2, and when $y=b$, $x=a$; when $x=b$, $y=-a$. Find x in terms of y.

(14) A spherical iron shell, 1 ft. in diameter, weighs $\frac{91}{216}$ of what it would weigh if solid; how thick is the metal?

(15) Planets attract their satellites with a force which at a given distance varies as the planet's mass; and for a given mass, inversely as the square of the distance. Also the square of a satellite's time of revolution varies directly as its distance from the planet and inversely as the force of the planet's attraction (the orbit being supposed circular). Show that if R represent the distance of a satellite from its primary, T the time of its revolution, and M the mass of its primary, $M \propto \frac{R^3}{T^2}$.

If for the moon and one of Jupiter's satellites the values of R are in the ratio 9 : 46; and of T 18 : 11, compare the mass of Jupiter with that of the earth.

NOTE ON CHAPTER XIX.

The treatment of Ratio has depended on the assumption that it is always possible to express the two quantities in *integers*. And this will be possible with a proper choice of unit, even when the quantities appear to have arithmetically a *fractional* form.

For instance, $2\frac{1}{2}$ in. and $4\frac{2}{3}$ in. are fractional in form, but if instead of an inch, the *sixth* of an inch be taken as the unit, the same lengths will be expressed by 15 and 28.

Two quantities which can be exactly expressed in terms of some common unit are called *commensurable*.

But two quantities may be compared which are *incommensurable*, *i.e.* such that no unit exists in terms of which *both* the quantities will be integers; as the side and diameter of a square. In this case it will be always possible, by taking the unit sufficiently small, to find a fraction which shall be as near as we please to the true value of the ratio, either above or below.

For instance, if a and b are the diameter and side of a square,

$a = b\sqrt{2}$, $\quad \therefore \; a : b :: \sqrt{2} : 1$.

Now $\sqrt{2} = 1{\cdot}41421709$, &c., which is $> 1{\cdot}414217$, but $< 1{\cdot}414218$.

If then we take *a millionth* as the unit,

$a : b$ lies between $1414217 : 1000000$, and $1414218 : 1000000$,

and differs from either by less than $\frac{1}{1000000}$.

By carrying the decimal further, we may obtain a still nearer approximation to the true value of $a : b$.

Expressed generally, if a and b are incommensurable, let b be divided into any integral number (n) of parts, and taking one of these as units, suppose it to be contained in a more than m times but less than $m+1$ times. Then

$$\frac{m}{n} < \frac{a}{b} \text{ and } \frac{m+1}{n} > \frac{a}{b}.$$

And the error therefore in taking either of these for $\frac{a}{b}$ is $< \frac{1}{n}$.

Now, by taking n large enough, we may make $\frac{1}{n}$ as small as we please, so that the error may be made less than any quantity

which may be assigned: any of the propositions which are proved above to be true for $\frac{m}{n}$ and $\frac{m+1}{n}$ may be therefore considered true for $\frac{a}{b}$.

If a second ratio $\frac{c}{d}$, where c and d are incommensurable, lie between the same limits, the difference $\frac{a}{b} \sim \frac{c}{d}$ is less than $\frac{1}{n}$, and is therefore less than any assignable difference;

$$\therefore \frac{a}{b} \text{ may be considered to be } = \frac{c}{d};$$

and the propositions on Proportion which are proved above will be true for a, b, c, d.

CHAPTER XX.

PROGRESSIONS.

(See also *Arithmetic*, ch. xvii.)

ARITHMETICAL PROGRESSION.

95.—A series of numbers, or quantities, which increase or decrease by a common difference is said to be in *Arithmetical Progression*: as

$$2,\ 5,\ 8,\ 11,\ 14,\ \&c.$$
$$30,\ 26,\ 22,\ 18,\ 14,\ \&c.$$

The general representative of such a series will be

$$a,\ a+d,\ a+2d,\ a+3d,\ \&c.$$

in which a is the first term, and d the common difference; and the series will be increasing or decreasing according as d is positive or negative.

Since each term of the series is obtained from the preceding one by adding d, it will be seen that the coefficient of d will be always 1 less than the number of the term; so that

$$n\text{th term} = a+(n-1)d \quad . \quad . \quad . \quad (1)$$

If the nth term be denoted by N, (1) may be regarded as an equation between a, d, n, N; from which, if any three of the quantities are given, the fourth may be found.

96.—The *arithmetical mean* between two numbers is the number which stands between them, and makes with them an arithmetical series.

If a and b are the two numbers, and A denote their arithmetical mean, we have by the definition,

$$A-a = b-A,$$
$$\therefore \quad A = \frac{a+b}{2} \quad . \quad . \quad . \quad . \quad (2)$$

Sometimes it is required to insert *several* arithmetical means between two numbers, *i.e.* to insert between them several numbers, which with them will form a continuous arithmetical series.

Ex. To insert four arithmetical means between 6 and 26.

We notice that 6, 26, and the four means form 6 terms, so that equation (1) gives $26 = 6 + 5d$

$$\therefore \ d = 4$$

and the means are 10, 14, 18, 22.

Prop. Prove *that if* r *means are to be inserted between* a *and* b, *the common difference will be* $\frac{b-a}{r+1}$, *and the means will be* $\frac{ra+b}{r+1}$, $\frac{(r-1)a+2b}{r+1}$, $\frac{(r-2)a+3b}{r+1}$, &c.

97.—Prop. *The sum of any two terms equidistant from the beginning and end of an arithmetical series is equal to the sum of the extreme terms.*

For if a and l be the first and last terms, the series may be written either as the increasing series

$$a, a+d, a+2d, \ldots \ldots \ l-d, l;$$

or as the decreasing series (in the reverse order)

$$l, l-d, l-2d, \ldots \ldots \ a+d, a.$$

The rth from the beginning will (from the upper line) be

$$a+(r-1)d;$$

and (from the lower line) the rth term from the end will be

$$l-(r-1)d;$$

and the sum of these two is $a+l$.

Prop. *To find the sum of all the terms of an arithmetical series.*

Let n be the number of terms, s their sum;

then $s = a+(a+d)+(a+2d)+ \ldots \ldots (l-d)+l;$

also $s = l+(l-d)+(l-2d)+ \ldots \ldots (a+d)+a;$

$\therefore$ adding these two lines,

$$2s = (a+l)+(a+l)+(a+l) \ \ldots \ n \text{ terms.}$$

$$= n\,(a+l)$$

$$\therefore \ s = \frac{n}{2}\,(a+l) \ \ldots \ldots \ldots \ldots \ (3)$$

Since by (1) $l=a+(n-1)d$, we have by substituting this value in (2) $s=\frac{n}{2}\{2a+(n-1)d\}$ . . . (4)

from which the sum of any given number of terms may be found, when the first term and common difference are given.

(4) is an equation between a, d, n, s, from which, if any three of these quantities are given, the fourth may be determined.

It will be noticed that the equation for finding n will be *quadratic*; and as in the case of other problems involving quadratics, sometimes both, but sometimes only one of the roots, will give solutions.

Exercise 44.

(1) Find the 13th term of $5, \quad 9, \quad 13, \ldots.$

" 7th " $12, \quad 9, \quad 6, \ldots.$

" 9th " $-3, -1, \quad 1, \ldots.$

" 10th " $-2, -5, \quad -8, \ldots.$

" 8th " $a, a+3b, a+6b, \ldots.$

" 5th " $a-b, a+b, \ldots.$

(2) Find the arithmetical means between 3 and 12; -5 and 17; a^2+ab-b^2 and a^2-ab+b^2.

(3) What term of the series whose first term is 2, and common difference $\frac{1}{8}$, will be 10?

(4) The 7th term of a series whose common difference is 3, is 11, find the first term.

(5) The 4th term of a series is 5, and the 10th is 12, what is the 6th term?

(6) Insert 3 arithmetical means between 1 and 19; and 4 between -3 and 17.

(7) Obtain the rth term of the series $\frac{x+3y}{2}$, $\frac{3x+y}{2}$, &c.

(8) Sum $5+8+11+\ .\ .\ .$ to 10 terms;
„ $7+5+3+\ .\ .\ .$ to 12 terms;
„ $-4-1+2+\ .\ .\ .$ to 7 terms;
„ $-3-8-13-\ .\ .\ .$ to 6 terms;
„ $a+4a+7a+\ .\ .\ .$ to n terms;
„ $(a-b)+(a+b)+\ .\ .\ .$ to 11 terms.

(9) The sum of six numbers in arithmetical progression is 27, and the first term is 1; determine the series.

(10) How many terms of $-5-2+1+\ .\ .\ .$ must be taken so that their sum may be 63?

(11) The first term is 12 and the sum of 10 terms is 10; find the last term.

(12) The arithmetical mean between two numbers is 10, and the mean between the double of the first and the treble of the second is 27: find the numbers.

(13) In an odd number of terms show that the sum of the first and last is double the middle term.

Find the middle of 11 terms whose sum is 66.

(14) The sum of three numbers in arithmetical progression is 15, and the sum of their squares is 83. Find them. (Take x for the *middle* number.)

(15) Arithmetical means are inserted between 5 and 23 such that the sum of the first two : sum of last two :: 2 : 5. How many means are there?

(16) How many terms of the series 1, 4, 7 . . . must be taken in order that the sum of the first half may bear to the sum of the second half, the ratio 10 : 31?

(17) A travels uniformly 20 miles a day; B starts 3 days later and travels 8 miles the first day, 12 the second, and so on in arithmetical progression. In how many days will B overtake A?

(18) Show that if any even number of terms of the series 1, 3, 5, . . . be taken, the sum of the first half : sum of the second half :: 1 : 3.

(19) The sum of five numbers in arithmetical progression is 45, and the product of the 1st and 5th is $\frac{5}{8}$ of that of the 2nd and 4th. What are the numbers?

(20) If a full truck descending an incline draws up an empty one at the rate of $1\frac{1}{2}$ ft. the first second, $4\frac{1}{2}$ ft. the second, $7\frac{1}{2}$ ft. the third, and so on; how long will it take on an incline 150 ft. in length? What part of the distance will it have descended in half the time?

GEOMETRICAL PROGRESSION.

98.—A series of numbers or quantities is said to be in *Geometrical Progression* when one is obtained by multiplying the preceding one by a constant factor; as

$$2, \ 6, \ 18, \ 54, \ \&c.$$
$$9, \ 6, \ 4, \ 2\tfrac{2}{3}, \ \&c.$$

The general representative of such a series will be

$$a, \ ar, \ ar^2, \ ar^3, \ \&c.$$

in which a is the first term, and r the factor or *common ratio.*

Since the index of r increases by 1 for every term, it will be seen that it will be always 1 less than the number of the term, so that

$$n\text{th term} = ar^{n-1} \quad . \quad . \quad . \quad . \quad (1)$$

99.—The *geometrical mean* between two numbers is the number which when placed between them will make with them a geometrical series.

If a and b are the two numbers, and G denote their geometrical mean, we have by the definition,

$$\frac{G}{a} = \frac{b}{G}$$

$$\therefore \ G = \sqrt{ab} \quad . \quad . \quad (2)$$

Thus if 8 and 18 are the two numbers, their mean will be $\sqrt{144}$, *i.e.* 12.

It will be noticed that the geometrical mean is identical with the *mean proportional* between two numbers.

Sometimes it may be required to insert *two or more* geometrical means between two numbers, *i.e.* to insert between them two or more numbers which will with them form a continuous geometrical series.

Thus *to insert two means between* 4 and 108; 108 *will be the fourth term of the series;*

$$\therefore\ 4r^3 = 108,$$
$$r^3 = 27, \text{ and } r = 3.$$

Hence the means are 12, 36.

PROP.—*The product of any two terms equidistant from the beginning and end of a geometrical series is equal to the product of the extreme terms.*

For if a and l be the first and last terms, the series may be obtained either by multiplying a successively by r, viz.,

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots$$

or (in reverse order) by dividing l successively by r, viz.,

$$l,\ \frac{l}{r},\ \frac{l}{r^2},\ \frac{l}{r^3},\ \ldots\ldots$$

The nth term from the beginning is therefore ar^{n-1}, and the nth from the end, $\frac{l}{r^{n-1}}$: and their product is al.

100.—PROP. *To find the sum of the terms of a geometrical series.*

Let n be the number of terms, s their sum; then

$$s = a + ar + ar^2 + \ \ldots\ + ar^{n-1}$$
$$\therefore\ rs = \quad ar + ar^2 + \ \ldots\ + ar^{n-1} + ar^n.$$

$\therefore$ subtracting the upper line from the lower

$$rs - s = ar^n - a,$$

or
$$(r-1)s = a(r^n - 1).$$

$$\therefore\ s = a \cdot \frac{r^n - 1}{r - 1} \qquad \ldots \qquad (3)$$

If $r < 1$, this formula will be more convenient for use if written as

$$s = a \cdot \frac{1 - r^n}{1 - r}$$

As in this formula n is an *index*, it will not be so easy in geometrical as in arithmetical progression, to find n or r in terms of the remaining letters.

Exercise 45.

(1) Find the 7th term of $2,\ 6,\ 18,\ \ldots\ldots$

„ 6th „ $3,\ 6,\ 12,\ \ldots\ldots$

„ 9th „ $6,\ 3,\ 1\frac{1}{2},\ \ldots\ldots$

„ 8th „ $1,\ -2,\ 4,\ \ldots\ldots$

„ 12th „ $x^3,\ x^4,\ x^5,\ \ldots\ldots$

„ 5th „ $4a,\ -6ma^2,\ 9m^2a^3,\ \ldots\ldots$

(2) Find the geometrical means between 5 and 20; $2\frac{1}{2}$ and $\frac{2}{5}$; $3ab^3$ and $27a^5b^5$; $18x^3y$ and $50xy^3z^2$.

(3) The fifth term of a series is 48 and the ratio 2; determine the first and seventh terms.

(4) Determine the ratio when the first term is 5 and the third 80.

(5) Insert two geometrical means between 8 and 125; and three between 14 and 224.

(6) If $a=2$, and $r=3$, which term will equal 162?

(7) A series whose ratio is $\frac{2x}{3}$ has $24x^3$ for its sixth term; what is its tenth term?

(8) Sum $3+6+12+$ . . . to 8 terms.
 " $1-3+9-$ . . . to 6 and to 7 terms.
 " $8+4+2+$ . . . to 10 terms.
 " $\cdot1+\cdot5+2\cdot5+$ . . . to 7 terms.
 " $m-\frac{m}{4}+\frac{m}{16}-$ . . . to 5 terms.
 " $x^4+x^3y+x^2y^2+$ . . . to 9 terms.

(9) The sum of four numbers in geometrical progression is 200, and the first term is 5; find the ratio.

(10) Find the sum of eight terms of a series whose last term is 1 and fifth term $\frac{1}{8}$.

(11) In an odd number of terms show that the product of the first and last will be equal to the square of the middle term.

(12) The product of four terms of a geometrical series is 4, and the fourth term is 4; determine the series.

(13) Of three numbers in geometrical progression the first and second exceed the third by 3, but the first and third exceed the second by 21; what are they?

(14) Show that if the terms of a geometrical series are taken in pairs, or in threes, the sums will be in geometrical progression: and determine the ratio in each case.

(15) If from a line one-third be cut off, then one-third of the remainder, and so on; what fraction of the whole will be left when this has been done five times?

What fraction will be left when $\frac{1}{m}$th has been cut off *in the same way n times?*

(16) A glass of wine is taken from a decanter which holds 10 glasses, and a glass of water poured in. When this has been done 5 times, how much wine is still left in the decanter?

(17) If from a vessel containing a gallons of wine, b gallons be taken out, and the vessel then filled up with water, and this process be repeated n times, how much wine will be left in, and how much water will have been taken out?

(18) Find two numbers whose sum is $3\frac{1}{4}$ and geometrical mean $1\frac{1}{2}$.

(19) The sum of the first half of a series is 63, and of the second 4032; and the first term is $\frac{3}{4}$ of the ratio. How many terms are there?

(20) If a, b, c, d are in geometrical progression, show that $(a-d)^2-(b-c)^2=(a-c)^2+(b-d)^2$.

101.—A geometrical series whose common ratio is less than 1, has its terms continually decreasing, and by taking n sufficiently large, the nth term, ar^{n-1}, may be made as small as we please. Also in the sum of n terms,

$$\frac{a(1-r^n)}{1-r} \text{ or } \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$\frac{ar^n}{1-r}$ becomes, when n is very large, indefinitely small; and the sum approaches therefore indefinitely near to $\frac{a}{1-r}$.

Thus the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ continually approaches $\frac{1}{1-\frac{1}{2}}$ *i.e.* 2, and by taking a sufficient number of terms may be made as near to 2 as we please.

$\frac{a}{1-r}$ is called the *Limit* of the sum of such a series, or its *sum to infinity*: it is often denoted by Σ.

Exercise 46.

(1) Find the sum to infinity of

$$3+1+\frac{1}{3}+\frac{1}{9}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$2+1\frac{1}{3}+\frac{8}{9}+\frac{16}{27}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\cdot1+\cdot01+\cdot001\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$4-\cdot8+\cdot16-\cdot032\ \cdot\ \cdot\ \cdot\ \cdot$$

(2) There is a series such that, if continued to infinity, either term is equal to the sum of all that follow: determine its ratio.

(3) The first two terms of a series are together equal to $5\frac{2}{5}$, and its sum to infinity is 15: determine the series.

HARMONICAL PROGRESSION.

102.—A series of numbers is said to be in *Harmonical Progression* when their reciprocals are in Arithmetical Progression.

Thus $1\frac{1}{2}$, 2, 3, 6 are in harmonical progression, since $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, are in arithmetical progression.

Generally, a, $a+d$, $a+2d$, . . . being in arithmetical, $\frac{1}{a}$, $\frac{1}{a+d}$, $\frac{1}{a+2d}$ . . . . will be in harmonical progression; and the nth term of the former being $a+(n-1)d$, the nth term of the latter will be $\frac{1}{a+(n-1)d}$.

Questions on harmonical series should be generally solved by inverting the given terms so as to obtain the corresponding arithmetical series, and then reinverting.

Ex. Insert three harmonical means between 3 and 19.

First find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{19}$:

$$\frac{1}{19}=\frac{1}{3}+4d,$$

$$\therefore\ d=-\frac{4}{57},$$

$\therefore$ the arithmetical means are $\frac{15}{57}$, $\frac{11}{57}$, $\frac{7}{57}$.

Consequently the harmonical means required will be $3\frac{4}{5}$, $5\frac{2}{11}$, $8\frac{1}{7}$.

103.—Prop. *If* a, b, c, *are in harmonical progression*, a : c :: a−b : b−c.

For $$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},$$

or $$\frac{a-b}{ab}=\frac{b-c}{bc},$$

$\therefore$ $$\frac{a-b}{b-c}=\frac{ab}{bc}=\frac{a}{c},$$

$\therefore$ $$a : c :: a-b : b-c.$$

104.—Prop. *To find the harmonical mean between two numbers.*

Let a and b be the numbers, and H their harmonical mean.

Then $$\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H},$$

$$\therefore \quad \frac{2}{H}=\frac{1}{a}+\frac{1}{b},$$

$$=\frac{a+b}{ab}.$$

$$\therefore \quad H=\frac{2ab}{a+b}.$$

There is no general expression for the *sum* of a series of numbers in harmonical progression.

Exercise 47.

(1) Find the 7th term of $3, 3\frac{3}{7}, 4, \ . \ . \ .$
12th . . $4, 4\frac{1}{6}, \ . \ . \ .$
nth . . $8, 9, \ . \ . \ .$
nth . . $a, b, \ . \ . \ .$

(2) Continue to two terms each way the harmonical series, two consecutive terms of which are 15, 16.

(3) Insert five harmonical means between 12 and 20; and m means between a and b.

(4) If p and q are the hours in which two men separately can do some work, show that the time they will

take together will be half of the harmonical mean between p and q.

(5) The first two terms of a harmonical series are 5 and 6; which term will equal 30?

(6) The 5th and 9th terms of a harmonical series are 8 and 12: obtain the first four terms.

(7) If $2p$ and $3q$ are the pth and qth terms of a harmonical series, show that the $(p+q)$th term will be $6(p-q)$.

(8) The difference between two numbers is 2, and their harmonical mean is $5\frac{5}{8}$: find them.

(9) Given any three numbers a, b, c: find a number which when added to each will produce a harmonical series.

(10) If a, b, c are in harmonical progression, show that $\frac{a}{b+c}$, $\frac{b}{a+c}$, $\frac{c}{a+b}$ are also in harmonical progression. Illustrate it by 2, 3, 6.

105.—Prop. Prove that *the geometrical mean between two numbers is also the geometrical mean between their arithmetical and harmonical means.*

106.—Prop. *To find the sum of the squares of numbers in arithmetical progression.*

Let $a, b, c, d, \ldots\ldots l$, be n numbers, δ their common difference;

then $\quad b=a+\delta,\ c=b+\delta,\ d=c+\delta$, &c.

Now
$$b^3=(a+\delta)^3=a^3+3a^2\delta+3a\delta^2+\delta^3,$$
$$c^3=(b+\delta)^3=b^3+3b^2\delta+3b\delta^2+\delta^3,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$l^3=(k+\delta)^3=k^3+3k^2\delta+3k\delta^2+\delta^3$$
$$(l+\delta)^3=l^3+3l^2\delta+3l\delta^2+\delta^3.$$

$\therefore$ adding all these lines together, and omitting $b^3, c^3, \ldots l^3$ (which will appear on both sides of the equation), we have

$$(l+\delta)^3=a^3+3(a^2+b^2\ \ldots\ l\ )\delta+3(a+b+\ \ldots\ l)\delta^2+n\delta^3$$
$$=a^3+3S\delta+3\frac{n}{2}(a+l)\delta^2+n\delta^3,$$

from which, *since* a, l, n, δ, are known, S can be immediately *found.*

If the series is $1, 2, 3, \ldots . n$, $\delta=1$,

and $$(n+1)^3=1+3S+3\frac{n}{2}(n+1)+n,$$

$$\therefore \quad 3S=n^3+\frac{3n^2}{2}+\frac{n}{2}.$$

$$=\frac{n}{2}(2n^2+3n+1).$$

$$=\frac{n}{2}(n+1)(2n+1).$$

$$\therefore \quad S=\frac{n(n+1)(2n+1)}{6}.$$

Prop. *Show that the sum of the cubes of the first* n *natural numbers is* $\left\{\frac{n}{2}(n+1)\right\}^2$.

107.—Prop. *To sum the series*

$$a+(a+d)r+(a+2d)r^2+(a+3d)r^3+ \ldots$$

in which each term is the product of corresponding terms in an arithmetical and geometrical series.

Let n be the number of terms,
s their sum;

then $s=a+(a+d)r+(a+2d)r^2+\ldots.+\{a+(n-1)d\}r^{n-1}$,

$\therefore\ rs= \qquad ar+(a+d)r^2+\ \ldots.+\{a+(n-2)d\}r^{n-1}$
$\qquad\qquad\qquad\qquad +\{a+(n-1)d\}r^n$.

$\therefore$ subtracting the lower line from the upper,

$$(1-r)s=a+dr+dr^2+dr^3+\ldots.dr^{n-1}-\{a+(n-1)d\}r^n,$$

$$=a+(dr+dr^2+\ldots.dr^{n-1}+dr^n)-(a+nd)r^n,$$

$$=a+\frac{dr(1-r^n)}{1-r}-(a+nd)r^n,$$

$$\therefore s=\frac{a(1-r^n)-ndr^n}{1-r}+\frac{dr(1-r^n)}{(1-r)^2}.$$

If $r<1$ we shall have the sum to infinity

$$=\frac{a}{1-r}+\frac{dr}{(1-r)^2}.$$

Ex. Sum the series $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}$ ..... to 6 terms, and to infinity.

Here $\qquad a=1,\ d=1,\ r=\frac{1}{2}.$

$$\therefore\ s=\frac{1-7\times\frac{1}{2^6}}{\frac{1}{2}}+\frac{\frac{1}{2}(1-\frac{1}{2^6})}{\frac{1}{4}},$$

$$=\frac{57}{32}+\frac{63}{32}.$$

$$=3\tfrac{3}{4},$$

$$\Sigma=\frac{1}{\frac{1}{4}}=4.$$

In obtaining the sum of any series formed in this way, it will be usually best to go through the process of the above proposition.

Exercise 48.

(1) Sum $1^2+3^2+5^2+$ ... to 12 terms and to n terms.

(2) Sum $4^2+7^2+10^2+$ .. to 10 terms.

(3) Sum $1^3+3^3+5^3+$ ... to 8 terms.

(4) Sum $x+2x^2+3x^3+$ .. to 10 terms.

(5) Sum $\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+$ ... to 7 terms and to infinity.

and $\qquad \frac{1}{3}-\frac{2}{3^2}+\frac{3}{3^3}-$ ... to 7 terms and to infinity.

CHAPTER XXI.

PERMUTATIONS AND COMBINATIONS.

108.—The *Permutations* of a number of letters or things are the different arrangements which can be made with them, with reference both to the particular letters, &c. taken, and the order in which they are taken.

Thus *ab*, *ba*, *ac*, *ca*, *bc*, *cb*, are permutations of the letters taken two and two together;

abc, *acb*, *bac*, &c., taken three and three together.

109.—PROP. *To find the number of permutations of* n *things taken two together.*

Let $a, b, c, \ldots l$ be n letters; then one letter a may be written before each of the other $n-1$ letters, giving $n-1$ permutations,

$$ab, ac, ad, \ldots al,$$

in which a stands first.

So there are $n-1$ permutations,

$$ba, bc, bd, \ldots bl,$$

in which b stands first; and similarly for each of the n letters.

And as these exhaust all the arrangements that can be made of the letters two together, the number on the whole will be

$$n(n-1) \ldots \ldots \ldots \ldots (1)$$

PROP. *To find the number of permutations of* n *things three together.*

$a, b, c, \ldots l$ being n letters, it will be seen that a pair ab may be written before each of the other $n-2$ letters, giving $n-2$ permutations,

$$abc, abd, \ldots abl,$$

in which ab stands first.

So there will be $n-2$ in which ba stands first,

$$bac, bad, \ldots bal;$$

and similarly for each other pair ac, ca, ad, &c.

Now by the proposition above, there are $n(n-1)$ of these different pairs, and as each gives rise to $n-2$ permutations three together, the total number of such permutations is

$$n(n-1)(n-2) \;.\;.\;.\;.\;.\;.\;.\;.\; (2)$$

Similarly it may be proved that there are

$$n(n-1)(n-2)(n-3)$$

permutations when the letters are taken four together.

It will be noticed that the expression contains in each case as many factors as there are things taken together, and it may be concluded from the character of the proof that a similar formula will hold, whatever be the number taken together.

But to show this more explicitly, the proof should be given in a general form:—

PROP. *To prove that the number of permutations of* n *things* r *together is*

$$n(n-1)(n-2) \;.\;.\;.\;.\;.\; (n-r+1) \;.\;.\;.\; (3)$$

Suppose that the formula holds when p things are taken together, *i. e.* that the number of permutations of n things p together is $n(n-1) \ldots (n-p+1)$; then we may prove that it holds when $(p+1)$ of the things are taken together.

For any one of these sets may be placed before each of the remaining $n-p$ letters, giving $n-p$ permutations of $(p+1)$ letters in which that set stands first; and as this may be done with every such set, the total number of permutations that will be formed will be

$$n(n-1) \;.\;.\;.\;.\; (n-p+1)(n-p),$$

i.e. the formula holds for $p+1$ together.

Now it has been proved above for the things taken 3 together; ∴ it is true when taken 4 together: ∴ being true for 4, it will be true for 5 together; ∴ for 6 together, &c., and generally for r together, whatever r may be.

N.B.—When *all* the things are taken in each permutation, r will equal n, and the formula $n(n-1) \ldots (n-r+1)$ will become

$$n(n-1) \;.\;.\;.\;.\;.\;.\; 1 \;.\;.\;.\;.\;.\;.\;.\; (4)$$

This last expression, which is the product of all the natural numbers from 1 up to n, is of frequent occurrence, and may *be written as* $\underline{/n}$; *it is called factorial* n.

110.—The *Combinations* of a number of letters or things are the different sets that can be made of them two or more together, without reference to their order in each set.

Thus ab, ba (which formed different permutations) are one combination of letters taken 2 together.

abc, abd, bcd . . . are combinations of letters taken 3 together.

111.—PROP. *The number of combinations that can be made with* n *things* r *together is*

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{1\,.\,2\,.\,3\ \ldots\ r}$$

For by (109 (4)) any combination of r letters will by varying the arrangement of the letters amongst themselves produce $\underline{|r}$ permutations.

$\therefore$ if C represent the number of combinations of n things r together,

$C\times\underline{|r}$ will be their number of permutations r together.

But the number of permutations $=n(n-1)\ldots(n-r+1)$;

$$\therefore C=\frac{n(n-1)\ \ldots\ (n-r+1)}{\underline{|r}}\quad\ldots\ldots\ (1)$$

If the numerator and denominator of the above expression be each multiplied by $(n-r)(n-r-1)\ \ldots\ 1$ or $\underline{|n-r}$,

it will become $\dfrac{n(n-1)\ \ldots\ (n-r+1)(n-r)\ \ldots\ 1}{\underline{|r}\,.\,\underline{|n-r}}$,

or $$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\ \ldots\ldots\ (2)$$

If r be interchanged with $n-r$, this formula remains unchanged; hence *the number of combinations of* n *things* r *together is equal to the number of combinations* n$-$r *together.*

This conclusion may be arrived at also by considering that if out of n things a set of r be taken together, a set of $n-r$ will be left; and any change in the former set will produce a corresponding change in the latter. Therefore the number of different ways in which the former set can be composed will be equal to the number of ways in which the latter can be composed.

The combinations of n things $n-r$ together are said to *be complementary* to those taken r together.

112.—PROP. *To find how many of* n *things must be taken together in order that the number of combinations formed may be the greatest.*

If $C_2, C_3, C_4, \ldots$ denote the number of combinations taken 2, 3, 4 .... together,

$$C_2 = \frac{n(n-1)}{1 \cdot 2}, \quad C_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, \text{ \&c.}$$

$$\therefore C_3 = C_2 \times \frac{n-2}{3}, \quad C_4 = C_3 \times \frac{n-3}{4}, \ldots C_{r+1} = C_r \times \frac{n-r}{r+1}.$$

Now the factors $\frac{n-2}{3}, \frac{n-3}{4}, \ldots$ become successively less (for their numerators decrease, and their denominators increase), and at some point become less than 1.

Also so long as the factors are greater than 1,

$C_2, C_3, C_4 \ldots$ are increasing,

but when the factors become less than 1, these numbers will begin to decrease.

Hence C_{r+1} will first become less than C_r (i.e. C_r will be greatest), when r is such that $\frac{n-r}{r+1}$ is first less than 1

i.e. when $r+1$ is first greater than $n-r$,

or (transposing) $\quad 2r > n-1$,

i.e. $\quad r > \frac{n-1}{2}$.

Now when n is even and equal to $2m$, r will be first greater than $\frac{2m-1}{2}$, when r is equal to m.

When n is odd, and equal to $2m+1$, r will be greater than $\frac{2m}{2}$, *i.e.* equal to $m+1$.

In this latter case we have also

$$\begin{aligned} C_{r-1} \left(\text{which } = C_r \times \frac{n-r+1}{r}\right), \\ &= C_r \times \frac{2m+1-(m+1)+1}{m+1}, \\ &= C_r. \end{aligned}$$

So that when the number of things is even, the number of combinations will be greatest *when half of the whole are taken together*; when the number of things is odd, there will be two equal numbers of combinations—*when the number of things taken together is just under and just over the half of the whole.*

113.—Prop. *To find the number of permutations that can be made of* n *letters taken all together, when* p *of them are alike.*

Let $a, a', a'', \ldots\ldots b, c, d,$ &c. be n letters, p of which are $a, a', a'', \ldots\ldots$

Then if all the permutations ($\underline{|n}$) were formed, they might be arranged in groups so that in each group $b, c, d, \ldots\ldots$ should have fixed places, while $a, a', a'', \ldots\ldots$ should be interchanged, as

$$abca'da'' \ldots\ldots$$
$$a'bcada'' \ldots\ldots$$
$$a''bca''da \ldots\ldots, \text{\&c.}$$

The number of permutations in each group would therefore be the number of interchanges that could be made of $a, a', a'', \ldots\ldots$ i.e. $\underline{|p}$;

and the number of groups would therefore be $\dfrac{\underline{|n}}{\underline{|p}}$.

Now suppose $a', a'', \ldots\ldots$ each to become a, then the permutations in any one group become all identical, and the total number of different permutations will become the same as the number of groups, i.e. $\dfrac{\underline{|n}}{\underline{|p}}$.

Similarly if q other letters are alike, r others alike, &c., the number of permutations that can be formed may be shown to be $\dfrac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}, \text{\&c.}}$.

Exercise 49.

(1) Write down in columns, according to the arrangement in (109), all the permutations of $a, b, c, d,$ taken two together, three together, all together.

(2) Of all the numbers that can be formed with four of the figures 5, 6, 7, 8, 9; how many will begin with 56? Write them all down.

(3) How many changes can be rung on five bells, all being rung in each change?

(4) With three consonants and two vowels, how many words of three letters can be formed beginning and ending with a consonant, and having a vowel for the middle letter?

(5) Out of twenty men, in how many different ways can four be chosen to be on guard? In how many of these would one particular man be taken, and from how many would he be left out?

(6) Of twelve books of the same size a shelf will hold five; how many different arrangements on the shelf may be made?

(7) Of eight men forming a boat's crew one is selected as stroke; how many arrangements of the rest are possible? When the four who row on each side are decided on, how many arrangements are still possible?

(8) In how many ways may an eleven be chosen out of eighteen cricketers?

(9) How many different signals can be made by hoisting six differently coloured flags one above the other, (1) when all are hoisted each time; (2) when any number of them may be hoisted at once?

(10) How many different signals can be made with six flags, two of which are red, three blue, and one white, all being hoisted for each signal?

(11) How many permutations can be made with all the letters of the words *garden*, *dimension*, *character*, *indeterminate*?

(12) From twelve soldiers and eight sailors, how many parties of three soldiers and two sailors could be formed?

(13) Find the number of combinations of 100 things 97 together.

(14) With twenty consonants and five vowels, how many different words may be formed consisting of three different consonants and two different vowels, any arrangement of letters being considered a word?

(15) How many different triangles will be formed by joining the angular points of an octagon, each triangle having its angular points at angular points of the figure?

What will be the *total* number of triangles made by the lines (produced indefinitely), *supposing that no two of the lines are parallel, and no three pass through the same point*?

(16) How many different numbers can be made with the digits 1, 2, 3, 4, 5, all together? How many times will each figure occur in each place? Determine the sum of all the numbers.

(17) Of thirty things, how many must be taken together in order that having that number for selection there may be the greatest possible variety of choice?

(18) There are m things of one kind and n of another; how many different sets could be made containing r of the first and s of the second?

(19) How many different throws can be made with four dice?

(20) In how many ways may ten persons be seated at a round table, so that in no two of the arrangements may every one have the same neighbours?

(21) The number of combinations of n things six together, being twice the number when taken five together, find n.

(22) Determine the sum of all the numbers that can be formed with all the digits 4, 7, 8, 9. Also the sum of all those that can be formed by taking two of them together.

(23) The number of combinations of n things r together is three times the number taken $r-1$ together, and half the number taken $r+1$ together: find n and r.

(24) In how many ways may 12 things be divided into three sets of 4?

(25) Show that the number of combinations of 7 things 5 together is the sum of the combinations of 6 things 4 and 5 together. Show that this is equivalent to the statement that the former are made up of those in which one particular thing is included and those from which it is excluded.

Prove that the numbers of combinations of n things r together and $r+1$ together are equal to the number when $n+1$ things are taken $r+1$ together.

CHAPTER XXII.

BINOMIAL THEOREM.

114.—The *Binomial Theorem* is a very important formula discovered by Newton, by means of which any power or root of a binomial expression may be readily obtained.

It is based upon the following proposition:—

115.—Prop. *To investigate the formation of the product of* n *binomial factors* $x+a$, $x+b$, $x+c$, &c.

By ordinary multiplication,

$$(x+a)(x+b)=x^2+(a+b)x+ab \quad \ldots\ldots\ldots\ldots \quad (1)$$

Multiply each side by $x+c$, then

$$\begin{aligned}(x+a)(x+b)(x+c)&=x^3+(a+b)x^2+abx\\ &\quad+\quad cx^2+(ac+bc)x+abc\\ \hline &=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc \quad (2)\end{aligned}$$

Now in the results (1) and (2) certain *laws* are observable:—

(i) The number of terms in each is one more than the number of factors on the left side.

(ii) The index of x is in the first term the same as the number of factors, and decreases by 1 in each succeeding term.

(iii) The coefficient of x in the first term is unity;
in the second term it is the sum of a, b, c, ...;
in the third, it is the sum of their products two and two,

ab, ac, bc;

the fourth term is the product of the three a, b, c.

And if one more factor is taken the same laws will hold;

for $\quad (x+a)(x+b)(x+c)(x+d)$

$$\begin{aligned}&=x^4+(a+b+c)x^3+(ab+ac+bc)x^2+abcx\\ &\quad+\quad dx^3+(ad+bd+cd)x^2+(abd+acd+bcd)x+abcd\\ \hline &=x^4+(a+b+c+d)x^3+(ab+ac+\,..\,+cd)x^2+(abc+\,..\,+bcd)x\\ &\qquad+abcd.\end{aligned}$$

It will be worth while to notice the coefficients of the two lines in this product:—

Of x^3, the coefficient in the upper line is the sum of a, b, c; in the lower the new term d.

Of x^2, in the upper line, the products of a, b, c, two and two; in the lower, the product of d with each one of these; making therefore together all the products of a, b, c, d, two and two.

Of x, in the upper line, the coefficient is the product of the three letters a, b, c; in the lower, the products of d with every pair of them; making therefore together all the products of a, b, c, d, three and three.

Will the same laws hold, whatever be the number of factors?

The proof will be of the same kind as that in Permutations:—

Suppose that the laws hold for r factors, so that

$$(x+a)(x+b)\ldots\ldots(x+h)=x^r+p_1x^{r-1}+p_2x^{r-2}+\ldots p_r,$$

where p_1 stands for the sum $a+b+\ldots+h$,

p_2 for $ab+ac+\ldots$ the sum of the products two and two;

p_3 for the sum of the products three and three;

p_r for the product of the r letters.

Multiplying by another factor $x+k$, the product of the $r+1$ factors becomes

$$\begin{array}{l} x^{r+1}+p_1x^r+px^{2r-1}+p_3x^{r-2}+\ldots+p_rx \\ \qquad +kx^r+p_1kx^{r-1}+p_2kx^{r-2}+\ldots+p_{r-1}kx+p_rk \\ \hline =x^{r+1}+(p_1+k)x^r+(p_2+p_1k)x^{r-1}+(p_3+p_2k)x^{r-2}+\ldots \\ \qquad\qquad +(p_r+p_{r-1}k)x+p_rk. \end{array}$$

Here the laws (i) and (ii) obviously hold; and as regards the coefficients,

$p_1+k\ =a+b+\ldots h+k=$ sum of the $r+1$ letters.

$p_2+p_1k=(ab+ac+\ldots)+(ak+bk+\ldots hk)$;

$=$ sum of *all* the products of the $(r+1)$ letters, two and two together.

$p_3+p_2k=(abc+abd+\ldots)+(abk+ack+\ldots)$

$=$ sum of *all* their products three and three together.

. . . .

$p_rk=abc\ldots hk=$ product of the $r+1$ letters.

All the laws therefore hold for $r+1$ factors, if they hold for r factors.

But they have been shown to hold for 4, therefore they will hold for any number of factors.

Note.—A proof of the kind employed above is called *Inductive.*

116.—Prop. *To deduce from the last proposition the Binomial Theorem; viz.*

$$(x+a)^n=$$
$$x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}+..+a^n.$$

It was proved that the product of n factors

$(x+a)(x+b)\ldots(x+l)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n.$

Now let $b=c=d=\ldots=l=a$; then the left-hand side of this equation becomes

$$(x+a)^n.$$

On the right-hand side,

$$p_1=a+a+\ldots=na.$$

In p_2 there are as many terms ab, ac, ... as can be formed by taking the n letters 2 together, viz. $\frac{n(n-1)}{1.2}$, and each term $=a^2$;

$$\therefore p_2=\frac{n(n-1)}{1.2}a^2.$$

Similarly p_3 contains $\frac{n(n-1)(n-2)}{1.2.3}$ terms, each equal to a^3;

$$\therefore p_3=\frac{n(n-1)(n-2)}{1.2.3}a^3.$$

&c.

$$\therefore (x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}$$
$$+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}+\ldots+a^n.$$

When the value of $(x+a)^n$ is obtained in this form it is said to be *expanded*, and the expression on the right-hand side is called the *expansion* of $(x+a)^n$.

Though in the proof it was convenient to obtain $(x+a)^n$,

yet in application it is frequently better to have a and x interchanged, so that the expansion may proceed by ascending powers of x, viz.—

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \ldots + nax^{n-1} + x^n.$$

If $a = 1$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots\ldots\ldots + nx^{n-1} + x^n.$$

In the proof, a and x have had any value: if x be negative, its odd powers will be negative and its even powers positive; and the above two lines become

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 - \ldots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1 \cdot 2}x^2 - \ldots\ldots\ldots$$

The last term in each line may be written as $(-1)^n x^n$, the symbol $(-1)^n$ being $+1$ when n is even, and -1 when n is odd.

The last term but one would in the same way have $(-1)^{n-1}$ prefixed to it.

The *number* of terms is obviously $n+1$.

Ex. (1) $$(a+x)^5 = a^5 + 5a^4x + \frac{5 \cdot 4}{1 \cdot 2}a^3x^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}a^2x^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}ax^4 + x^5.$$
$$= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

Ex. (2) $$(2x-3y)^4 = (2x)^4 - 4(2x)^3 3y + \frac{4 \cdot 3}{1 \cdot 2}(2x)^2(3y)^2 - \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}(2x)(3y)^3 + (3y)^4.$$
$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$$

The rth (or general) term in the expansion of $(a+x)^n$ will be $\dfrac{n(n-1)\ldots(n-r+2)}{1 \cdot 2 \ldots (r-1)}a^{n-r+1}x^{r-1}$

Note.—It will be convenient for application to observe that the factors in the numerator and denominator of the *coefficient* are the same in number; and that the *corresponding factors* have their sum $= n+1$. Also the last

factor of the denominator is the same as the index of x, and is 1 less than the number of the term. The indices of a and x together $=n$.

Ex. (3) The 5th term of $(x+y)^{12}$ will be

$$\frac{12.11.10.9}{1.2.3.4}x^8y^4,$$
$$=495x^8y^4.$$

117.—PROP. *Prove that the coefficient of the* r*th term from the beginning is equal to the coefficient of the* r*th term from the end; or the* (n−r+2)*th from the beginning. See* (111).

118.—PROP. *Find the greatest coefficient in the expansion of* $(a+x)^n$. *See* (112).

119.—If in the expansion of $(1+x)^n$ we put $x=1$, we obtain

$$2^n=1+n+\frac{n(n-1)}{1.2}+\ .\ .\ .\ .\ .$$

i.e. *the sum of all the coefficients* $=2^n$.

If we put $x=1$ in the expansion of $(1-x)^n$ we obtain

$$0=1-n+\frac{n(n-1)}{1.2}-\ .\ .\ .\ .$$

= sum of odd coefficients − sum of even coefficients.

∴ sum of odd coefficients = sum of even coefficients, and as both together $=2^n$, each sum $=\frac{1}{2}$ of 2^n, *i.e.* 2^{n-1}.

Exercise 50.

(1) In the product $(x+1)(x+2)(x+3)(x+4)$ find the coefficient of x^2; also in $(x+1)(x-2)(x-3)(x+4)$.

Expand

(2) $(a+x)^4$. (3) $(1+x)^7$. (4) $(a-x)^6$.

(5) $(1+2x)^5$. (6) $(x-3)^8$. (7) $(2x-3y)^4$.

(8) $\left(1-\frac{x}{3}\right)^9$ (9) $\left(1-\frac{3y}{4}\right)^5$

(10) Find the 3rd term of $(a+x)^{20}$.

(11) The 4th term of $(2x-5y)^{12}$.

(12) The 7th of $\left(\frac{x}{2}+\frac{y}{3}\right)^{10}$.

(13) The 12th of $(a^2-ax)^{15}$.

(14) The 8th of $(5x^2y-2xy^2)^9$.

(15) The middle term of $\left(\frac{x}{y}+\frac{y}{x}\right)^8$.

(16) The middle term of $\left(\frac{x}{y}-\frac{y}{x}\right)^{10}$.

(17) The two middle terms of $\left(\frac{x}{y}-\frac{y}{x}\right)^7$.

(18) The rth from the beginning and the rth from the end of $(2a+x)^n$.

(19) The $(r+4)$th of $(a-x)^n$.

(20) The middle term of $(a+x)^{2n}$.

(21) The two middle terms of $(a-x)^{2n+1}$.

(22) In the expansion of $(a+x)^n$ the coefficient of the 5th term is $\frac{8}{5}$ of that of the 4th; find n.

(23) Find the sum of the coefficients of $(a+x)^{12}$; and of $(2a+x)^{12}$. And the sum of the terms when $a=1$, $x=-2$.

(24) Four consecutive terms in the expansion of $(a+x)^n$ become 22, 220, 1320, 5280, when certain numerical values are substituted for a, x, n. What are these values?

(25) If A be the sum of the odd terms, and B of the even terms in the expansion of $(a+x)^n$, show that

$$A^2-B^2=(a^2-x^2)^n.$$

CHAPTER XXIII.

NOTATION.

(*Arithmetic*, chap. xxi.)

120.—*Notation* is the method of expressing numbers of any magnitude by means of a small number of digits. One number is chosen as the *base* or *radix*, and any number may then be expressed in powers of this, with the help of digits for 0 and all numbers less than the base. In the common system 10 is taken as the base, but any other base, 2, 3, 4 . . . r, might be adopted, and numbers expressed in that *scale*, as it is termed.

The process of representing a large number by means of digits may be illustrated in this way. Suppose a heap of corn to be taken, and it is required to express the number of grains in the common scale.

Count the grains first in 10's, giving (suppose)

q_1 heaps of 10 each, with a grains left ($a < 10$).

Put these heaps together in 10's, giving

q_2 heaps of 100 grains, with b small heaps left ($b < 10$).

Put these together in 10's, giving

q_3 heaps of 1000, with c of the second heaps left;

and so on, until the number of largest heaps becomes l where $l < 10$.

Then the total number of grains will be

$$a + b \text{ heaps of } 10 + c \text{ heaps of } 100 + \text{\&c.}$$

$$= a + 10b + 10^2c + 10^3d + \text{\&c.} \ldots + 10^n l.$$

This, according to the ordinary convention, would be represented arithmetically by $l \ldots . dbca$.

The same process is expressed algebraically in the following proposition.

121.—Prop. *To express any integral number N in the scale of r.*

Divide N by r, with quotient q_1 and remainder $a<r$;
Divide q_1 by r, with quotient q_2 and remainder $b<r$;
and so on; until a quotient q_n is obtained $<r$; let $q_n=l$.

$$
\begin{aligned}
\text{Then} \qquad N &= q_1r+a, \\
&= (q_2r+b)r+a, \\
&= q_2r^2+br+a, \\
&= \&c. \\
&= (q_nr^{n-1}+k)r+ \; \ldots \; +br+a, \\
&= lr^n+kr^{n-1}+ \; \ldots\ldots \; +cr^2+br+a.
\end{aligned}
$$

To express therefore any number in a proposed scale, *divide the number by the base, then the quotient by the base, and so on; the successive remainders will be the successive digits, beginning from the unit's place:* and this method may be applied, in whatever manner the number may be at first expressed: *e.g.* to express in any proposed scale a number given in the common scale, or to transform a number from one scale to another.

The number of digits will be 1 more than the highest index of r.

122.—Prop. *If two numbers containing respectively* p *and* q *digits be multiplied together, their product will contain either* p+q *or* p+q−1 *digits.*

For if $\qquad a+br+\ldots\ldots+lr^{p-1}$
and $\qquad \alpha+\beta r+\ldots\ldots+\lambda r^{q-1}$
be the numbers, the highest power of r in their product will be contained in $l\lambda r^{p+q-2}$.

Now l and λ are each >0, but $<r$,

$\therefore$ their product will be >0, but $<r^2$.

$\therefore$ $l\lambda r^{p+q-2}$ will contain as the highest power of r either r^{p+q-2} or r^{p+q-1},

and the number of digits will consequently be

$$p+q-1 \text{ or } p+q.$$

123.—Just as in the decimal scale it is convenient to express quantities which are less than 1 by means of decimal fractions, so *radix fractions* may be employed in any other scale; *i. e.* fractions whose denominators are successive powers of the radix.

Prop. *To express a quantity less than* 1 *in a series of radix fractions.*

Let M be the quantity, and let $\alpha, \beta, \gamma, \ldots.$ be the numbers; then

$$M=\frac{\alpha}{r}+\frac{\beta}{r^2}+\frac{\gamma}{r^3}+\ \ldots.$$

$\therefore$ multiplying by r,

$$Mr=\alpha+\frac{\beta}{r}+\frac{\gamma}{r^2}+\ \ldots.$$

So that α is the integer obtained in the product. Multiply the remainder by r, and the resulting integer will be β.

Similarly $\gamma, \ldots$ will be obtained.

Exercise 51.

(1) 6, 7, 8, 3, 2, being the digits of a number in the scale of r, beginning from the right, write down the algebraical value of the number.

Also when 0, 2, 0, 0, 7, 6 are the digits.

(2) Multiply together 234 and 125, where r is the base of the scale.

(3) Express 42897 and 286010 (common scale) in the scales of 6 and 8.

(4) Express $3t64e9$ (in the scale of 12) in the common scale.

(N.B. t, e, stand for 10, 11.)

(5) Express 593 in powers of 4; and 10000 in powers of 9.

(6) 140 (decimal scale) is represented in another scale as 352; find the radix of the latter.

(7) Explain the principle of *carrying* figures in addition, by the example of $2763+9339$ in the ordinary scale; and by $a+br+cr^2+\ \ldots$ and $l+mr+nr^2+\ \ldots$ in the scale of r.

(8) Multiply 3674 by 7216 in the scale of 8; and divide 452132 by 4 in the scale of 6.

(9) Transform 37214 from the octonary (scale of 8) to the nonary scale.

(10) Obtain the greatest and least numbers of 4 digits in the scales of 10 and r; and the greatest and least with n digits in the latter scale.

(11) If P, a number of p digits, be exactly divisible by Q, a number of q digits, how many digits will there be in the quotient?

(Let x be the number, and deduce from (122).)

(12) In what scale will the ordinary number 756 be expressed by 530?

(13) In what scale will 540 be the square of 23?

(14) How many digits will the square of a number of p digits in any scale contain? And if the given number be a perfect square, how many digits will there be in its square root?

(15) Show that 1234321 will in any scale be a perfect square, and obtain its square root.

(16) How many of the weights $8, 8^2, 8^3, \ldots$ lbs. will be required to weigh 5 tons, the smallest number possible being used?

(17) Express $\frac{5}{8}$ and ·0625 in a series of radix fractions in the scale of 4.

(18) Multiply 31·24 by ·31 in the scale of 5; and express $\frac{5}{8}$ in radix fractions in that scale.

(19) In what scale will 212, 1101, 1220 be in arithmetical progression?

(20) How many digits will the cube of a number of p digits contain? and how many will its cube root contain, supposing it to be an exact cube?

MISCELLANEOUS EXAMPLES.

Exercise 52.

(1) Extract the square root of

$$25x^4-30ax^3+49a^2x^2-24a^3x+16a^4.$$

(2) Solve the equations:—

(i) $(2x-3)^2=8x$.

(ii) $\dfrac{2x}{15}+\dfrac{3x-50}{3(10+x)}=\dfrac{12x+70}{190}$.

(iii) $x^2+y^2=50$, $x+y=8$.

(3) A farmer wishes to enclose a rectangular piece of land, to contain 1 acre 32 perches, with 176 hurdles, each 2 yards long; how many must he place in each side of the rectangle?

(4) What are eggs a dozen when two more in a shilling's worth lowers the price a penny a dozen?

(5) When is a series of numbers said to be in arithmetical or in geometrical progression? Find the last term and the sum of the series:—

(i) $5+1\frac{1}{2}-2-$ &c. to 20 terms.

(ii) $3+3\frac{1}{2}+4\frac{1}{12}+$ &c. to 6 terms.

(iii) $12-4+1\frac{1}{3}-$ &c. to 8 terms, and to infinity.

(6) Solve

$$\frac{4}{x+1}+\frac{3}{y-1}=\frac{1}{2}.$$

$$\frac{7}{x-1}-\frac{9}{y+1}=\frac{1}{4}.$$

(7) Find a number of two digits such that it is equal to twice the square of the first digit added to that of the second, and is also one less than five times the sum of the digits.

(8) Sum the series:—

(i) $5+9+13+$ &c. to 10 terms.

(ii) $5+7\frac{1}{2}+11\frac{1}{4}+$ &c. to 8 terms.

(iii) $2-1\frac{1}{5}+\frac{18}{25}-$ &c. to 6 terms, and to infinity.

(9) There are three numbers in harmonical progression, of which the third is double the first, and the second is ten less than the sum of the first and third. Find the numbers.

(10) Two trains travel at equal rates along a railway in opposite directions, starting at eight o'clock and half-past eight respectively; and they pass one another at a quarter before eleven. If their rates were increased by 10 miles an hour, and the latter started six minutes earlier, they would meet at the same point as before. Find the length of the line.

(11) Solve

(i) $x-y=xy=3x+4y$.

(ii) $x^2-y^2=xy=3x+4y$.

(iii) $9x+16y=250$ in positive integers.

(12) The number of yards of carpet required for a room is two more than double the number of inches in its width; but if the width were diminished by 3 inches, 7 yards more would be wanted. Find the length of the room, its breadth being 18 feet.

(13) Extract the cube root of

$$x^6-6x^5y+40x^3y^3-96xy^5-64y^6.$$

(14) Sum the series

(i) $\frac{5}{6}+\frac{3}{2}+\frac{13}{6}+\ldots$ to 17 terms;

(ii) $5\frac{3}{4}+4\frac{3}{5}+3\frac{17}{25}+\ldots$ to 7 terms, and to infinity;

and find the 22nd term of the harmonic series 30, 35, 42, ...

(15) A, who receives 6*d*. the first month, has his wages doubled every month for a year. B receives 6*d*. the first month, and has his wages increased each month in arithmetical progression for the same time. In the last month B's wages are 11*s*. less than half of A's; how much does one receive more than the other in the course of the year?

(16) Show that

$$\frac{1}{(x-y)(x-z)(a+x)}+\frac{1}{(y-x)(y-z)(a+y)}$$
$$+\frac{1}{(z-x)(z-y)(a+z)}=\frac{1}{(a+x)(a+y)(a+z)}.$$

(17) Find the number of permutations, and of combinations, which can be made with 13 things 6 together.

(18) Two numbers are in the ratio $8 : 11$, and when 10 is added to each, they are in the ratio $7 : 9$; what are they?

(19) Expand $(y-5)^6$.

(20) Transform 27932 from the undenary to the duodenary scale.

(21) If $x \propto y$ and $y \propto z$, and when $x=4$, $y=5$; and when $y=7$, $z=8$; find x in terms of z.

(22) Find the cube root of

$$64+240x+348x^2+245x^3+87x^4+15x^5+x^6.$$

(23) Solve

$$\frac{3x-2}{x+1}-\frac{5}{6}\cdot\frac{x-2}{3x+1}=\frac{5}{3}.$$

(24) 4 and 6 are the first two terms of a progression; find the 8th term, when the series is (i) arithmetical, (ii) geometrical, (iii) harmonical.

(25) If the 26 letters of the alphabet were combined 8 together, in how many of the combinations would w, x, y, z occur?

(26) Transfer $23 \cdot 264$ from the denary to the quinary scale.

(27) The arithmetical and geometrical means between two numbers being a and g, find their harmonical mean.

(28) Solve $6x+11y=47$

(i) in positive integers;

(ii) so that y may be a multiple of x, and positive.

(29) In how many ways may 5 sentries be chosen from 24 men? and in how many ways may their posts be allotted to each 5?

(30) Sum $1+3x+5x^2+7x^3+ \ldots$ to 12 terms.

(31) A and B set out at the same time; A from P to go to Q, and B from Q to go to P; they meet on the road when A has travelled 30 miles more than B, and find that if A continues at the same rate, he will reach Q in 4 days, and that B at his rate will reach P in 9 days. Find the distance P Q.

(32) If $y=p+q+r$, where p is constant, $q \propto x$, and $r \propto \frac{1}{x}$, and if when $x=1, 2, 3$, $y=3, 5\frac{1}{2}, 7$; find y when $x=4$.

(33) Solve $144x^4-73x^2+4=0$.

(34) If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{pa^2-qac+rc^2}{pb^2-qbd+rd^2}=\frac{a^2}{b^2}$.

(35) The fourth term of an arithmetical series being 36, and the seventh 57, find the first term and the common difference.

(36) Solve $x^4-4x^3-14x^2+36x+45=0$.

(37) Solve $1-4x-14x^2+36x^3+45x^4=0$.

(38) The sum of an infinite geometrical series is S, and the sum of the squares of the terms $=s^2$; find the first term and the ratio.

(39) A person travels part of a journey by rail, and the rest by coach; and observes that if the rate of the train is 20 miles an hour, and of the coach 8, he takes $3\frac{1}{2}$ hours altogether; but if the rates are 30 and 9 miles, he takes 2 hours 50 min. How long would he take to travel the whole by rail at the rate of 25 miles an hour?

(40) Insert 3 geometrical means between 16 and 625.

(41) Solve $(x-3)(x+4)(x-5)=0$,
and $(x^2-3x-28)(x^2-2x-3)=0$.

(42) There are 3 companies of soldiers, each consisting of 20 men; in how many ways may 3 men be chosen, 1 from each company?

(43) How many different signals might be made by hoisting flags of the same colour on the three masts of a ship, each mast being capable of having 5 flags hoisted at once.

(44) Solve $5x+6xy+5y=-8$.
$2x-3xy+2y=13$.

(45) In what scale will 72 be expressed by 242? and in what scale will ·84 be expressed by ·41?

(46) Show that the total number of combinations that can be made with n things, $1, 2, 3 \ldots n$ together is 2^n-1. (*See* (119).)

How many different sums can be formed with a *farthing*, a penny, a sixpence, a shilling, a half-crown, a *crown*, a half-sovereign, and a sovereign?

(47) Expand

$$\left(2x+\frac{y}{2}\right)^3-\left(2x-\frac{y}{2}\right)^3; \text{ and}$$

$$\left(2x+\frac{y}{2}\right)^6-\left(2x-\frac{y}{2}\right)^6;$$

and show that the latter contains the former.

(48) If the odd numbers are arranged in groups, 1 in the first group, 2 in the second, 3 in the third, and so on; show that the sum of the numbers in any group equals the cube of the number in the group.

(49) If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}=$ &c., then $a, b, c, d, \ldots$ will be in geometrical progression.

(50) If m shillings in a row reach as far as n sovereigns, and a pile of p shillings is as high as a pile of q sovereigns; on the supposition that the coins are perfectly smooth, compare the values of equal bulks of gold and silver.

CHAPTER XXIV.

SURDS. NEGATIVE AND FRACTIONAL INDICES.

124.—A *Surd* or *Irrational Quantity* is a root which cannot be exactly obtained, as $\sqrt{3}$, $\sqrt[3]{a}$, $\sqrt{a+b}$. The root which is required indicates what may be called the *order* of the surd; and surds are named according to their order, viz. *quadratic*, when the square root is required; *cubic*, when the cube root; *biquadratic*, when the fourth root is required.

The product of a rational and a surd factor is called a *mixed* surd, as $a\sqrt{b}$; when there is no rational factor outside the radical sign, the surd is said to be *entire*, as $\sqrt[3]{a^2b}$.

125.—Since any root of a quantity may be obtained (60) by taking that root of each of its factors, we have

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

and $$\sqrt[m]{ab} = \sqrt[m]{a} \times \sqrt[m]{b}.$$

Hence inversely,

$$\sqrt[m]{a} \times \sqrt[m]{b} \text{ will equal } \sqrt[m]{ab},$$

i.e. *the product of two surds of the same order will be the root of the product of the numbers which are under the radical signs.*

Similarly $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$,
i.e. *a factor under the radical sign, whose root can be taken, may by having the root taken be removed from under the radical sign.*

And inversely, *a factor outside the radical sign may be placed under it by being raised to the corresponding power;* thus

$$3\sqrt{2} = \sqrt{18};\ a\sqrt[3]{ab} = \sqrt[3]{a^4b}.$$

Also $\quad \sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}},$

$$\sqrt{\frac{a}{b^2}} = \frac{\sqrt{a}}{\sqrt{b^2}} = \frac{1}{b}\sqrt{a},$$

$$\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \frac{1}{b}\sqrt{ab}.$$

A surd is considered to be in its simplest form when the quantity under the radical sign is integral and as small as possible. The simplification will be effected by the above methods.

Ex. $\quad \sqrt{20} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}.$

$$\sqrt[3]{54} = \sqrt[3]{27} \times \sqrt[3]{2} = 3\sqrt[3]{2}.$$

$$\sqrt[3]{5x^2y^4} = y\sqrt[3]{5x^2y}.$$

$$\sqrt{\frac{5}{12}} = \sqrt{\frac{5}{4\times 3}} = \sqrt{\frac{15}{4\times 9}} = \frac{1}{6}\sqrt{15}.$$

$$\sqrt[4]{\frac{3x}{2y^3z^2}} = \sqrt[4]{\frac{24xyz^2}{16y^4z^4}} = \frac{1}{2yz}\sqrt[4]{24xyz^2}.$$

Surds which after being simplified have the same surd factor are said to be *similar*.

Exercise 53.

(*Arithmetic*, Ex. 89, 90.)

(1) Express as entire surds $5\sqrt{3}$, $a^2b\sqrt{bc}$, $x\sqrt[3]{x^2y^2}$ $3y^2\sqrt[4]{x^3y}$.

(2) Express as mixed surds $\sqrt{x^2y^4z}$, $\sqrt{8a^3b}$, $\sqrt[3]{54a^4x^2y^3}$, $2\sqrt[4]{80a^5b^2c^6}$.

(3) Simplify $2\sqrt{5} \times 3\sqrt{2}$, $7\sqrt[3]{4} \times 2\sqrt[3]{3}$, $\sqrt{20} \times \sqrt{15}$, $5\sqrt[4]{30} \times 2\sqrt[4]{48}$.

(4) Simplify $a\sqrt{x} \times b\sqrt{y}$, $\sqrt{ax} \times \sqrt{bx}$, $\sqrt[3]{2a^2b^4} \times \sqrt[3]{b^2x^3}$, $5\sqrt[4]{3a^3b^4y} \times \sqrt[4]{a^5by^3}$.

(5) Simplify $\sqrt{\dfrac{a}{2}}$, $\sqrt[3]{\dfrac{2xy^2}{x}}$, $\sqrt[3]{\dfrac{4}{25}}$, $\dfrac{a}{b}\sqrt[4]{\dfrac{b}{2a^3}}$, $\sqrt{\dfrac{3a^2bx}{4cy^3}}$.

(6) Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{4}{5}}$ are similar surds; and also $2\sqrt[3]{a^3b^2}$, $\sqrt[3]{8b^5}$, $\frac{1}{2}\sqrt[3]{\frac{a^6}{b}}$.

126.—The order of a surd may be changed, by changing the power of the quantity under the radical sign.

Thus $\sqrt[3]{a}=\sqrt[6]{a^2}$; $\sqrt{5}=\sqrt[4]{25}$: so conversely, as in (67),

$\sqrt[6]{a^2}$ may be reduced to $\sqrt[3]{a}$,

$\sqrt[4]{x^6y^2}$ may be reduced to $\sqrt{x^3y}$;

or expressed generally $\sqrt[mn]{a^m} = \sqrt[n]{a}$.

By this means surds of different orders may be reduced to the same order, and may then be compared, multiplied, or divided.

Ex. (1) To compare $\sqrt{2}$ and $\sqrt[3]{3}$.

$$\sqrt{2}=\sqrt[6]{8}; \text{ and } \sqrt[3]{3}=\sqrt[6]{9}.$$

$$\therefore \sqrt[3]{3} \text{ is the greater.}$$

Ex. (2) To multiply $\sqrt[3]{4a}$ by $\sqrt{6x}$.

$$\begin{aligned}\sqrt[3]{4a}\times\sqrt{6x} &= \sqrt[6]{16a^2}\times\sqrt[6]{216x^3},\\ &=\sqrt[6]{16a^2\times 216x^3},\\ &=\sqrt[6]{64\times 54a^2x^3},\\ &=2\sqrt[6]{54a^2x^3}.\end{aligned}$$

Exercise 54.

(1) Write in order of magnitude $2\sqrt[3]{3}$, $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{4}$.

(2) Simplify

$2\sqrt{ax}\times\sqrt[3]{3a^2b}\times\sqrt{2bx}$, and $\sqrt[4]{a^3xy^3}\times\sqrt[5]{a^2xy}$.

(3) Simplify

$3\sqrt[3]{4ab^2}\div\sqrt{2a^3b}$, and $\sqrt[4]{2a^3b^2}\times\sqrt[3]{a^5b^3}\div\sqrt{a^3b^5}$.

127.—The sum or difference (when unequal) of two *similar* surds will be a surd; for $m\sqrt{x}\pm n\sqrt{x}=(m\pm n)\sqrt{x}$: but their product or quotient will be rational, for

$$m\sqrt{x}\times n\sqrt{x}=mnx;\ \frac{m\sqrt{x}}{n\sqrt{x}}=\frac{m}{n}.$$

128.—**Prop.** *The product or quotient of two dissimilar quadratic surds will be a quadratic surd.*

For every quadratic surd when simplified will have under the radical sign a series of factors, each of which is raised only to the first power; and two surds which are dissimilar cannot have *all* these factors alike.

Hence their product or quotient will have at least one factor raised only to the first power, and will therefore be a surd.

129.—**Prop.** *The sum of two dissimilar quadratic surds cannot be a rational number; nor can it be expressed as a single surd.*

For if $\sqrt{a}+\sqrt{b}$ could equal a rational number c, we should have by squaring,

$$a+2\sqrt{ab}+b=c^2.$$

$$\therefore\ 2\sqrt{ab}=c^2-a-b.$$

Now c^2, a, b, being all rational, the right-hand side of this equation would be rational, whereas by the last proposition $\sqrt{ab}$ cannot be rational.

Hence $\sqrt{a}+\sqrt{b}$ cannot be rational.

Similarly it may be proved that $\sqrt{a}+\sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

Similarly, *the difference of two dissimilar quadratic surds is not rational, nor expressible as a single surd.*

130.—**Prop.** Prove as above that *a quadratic surd cannot equal the sum of a rational number and a surd.*

131.—*Compound* expressions involving surds will be *multiplied* together by taking the products of the several terms.

Thus

$$\begin{aligned}(2\sqrt{3}+5\sqrt{2})\times(5\sqrt{3}-2\sqrt{2}) &= 30+25\sqrt{6}-4\sqrt{6}-20\\ &= 10+21\sqrt{6}.\end{aligned}$$

To *divide* by a compound surd, both divisor and dividend must be multiplied by a factor which will *rationalise* the divisor.

Such a factor will be found in this way:—If the expression to be rationalised be binomial, and involve only quadratic surds, the factor will consist of the same terms but with a different sign between them.

Thus $6+2\sqrt{5}$ will be rationalised by the factor $6-2\sqrt{5}$; for $(6+2\sqrt{5})(6-2\sqrt{5}) = 36-20 = 16$.

If the expression be trinomial, two steps will be required.

Thus

$\sqrt{6}+\sqrt{3}-\sqrt{2}$ when multiplied by $\sqrt{6}-(\sqrt{3}-\sqrt{2})$ becomes $6-(\sqrt{3}-\sqrt{2})^2 = 6-5+2\sqrt{6} = 1+2\sqrt{6}$.

And $\qquad 1+2\sqrt{6}$ multiplied by $1-2\sqrt{6}$

becomes $\qquad 1-24 = -23$.

The process for division will then be as in the following example:—

$$\frac{7-3\sqrt{5}}{6+2\sqrt{5}} = \frac{(7-3\sqrt{5})(6-2\sqrt{5})}{(6+2\sqrt{5})(6-2\sqrt{5})} = \frac{72-32\sqrt{5}}{16} = \frac{9}{2}-2\sqrt{5}$$

The factor which will rationalise compound surds of a higher order will be longer; the method to be adopted will be seen in this example:—

To rationalise $2\sqrt[3]{2}-\sqrt[3]{5}$.

The surds here are both cubic.

We know that x^3-y^3 is divisible by $x-y$ with quotient x^2+xy+y^2;

$\therefore$ $x-y$ when multiplied by x^2+xy+y^2 will become x^3-y^3.

Hence the above will become $(2\sqrt[3]{2})^3-(\sqrt[3]{5})^3$, *i. e.* $16-5 = 11$, if multiplied by $(2\sqrt[3]{2})^2+2\sqrt[3]{2}\cdot\sqrt[3]{5}+(\sqrt[3]{5})^2$
$$= 4\sqrt[3]{4}+2\sqrt[3]{10}+\sqrt[3]{25}.$$

132.—Prop. *If* $a+\sqrt{b} = x+\sqrt{y}$, *then* a *will equal* x, *and* b *will equal* y.

For by transposing, $\sqrt{b}-\sqrt{y}=x-a$, and if b were not equal to y, the difference of two unequal surds would here be rational, which has been proved to be impossible.

$$\therefore \; b=y, \text{ and } a=x.$$

Similarly *if* $a-\sqrt{b}=x-\sqrt{y}$, a *will equal* x, *and* b *will equal* y.

N.B.—The above propositions have for simplicity been *proved of quadratic* surds, but they might be extended to *surds of higher orders.*

133.—*To extract the square root of a binomial surd,* $a+\sqrt{b}$.

Let $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ . . . . . (1)

then $a+\sqrt{b}=x+2\sqrt{xy}+y$.

$\therefore$ (131) $x+y=a$,

$2\sqrt{xy}=\sqrt{b}$,

which are two equations for finding x and y.

The above method may be shortened by noticing that, since $\sqrt{b}=2\sqrt{xy}$,

$$a-\sqrt{b}=x-2\sqrt{xy}+y,$$

$$\therefore \quad \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y} \quad . \; . \; . \; . \; (2)$$

Hence, multiplying (1) and (2),

$$x-y=\sqrt{a^2-b}.$$

And as $x+y=a$,

x and y will be found at once by addition and subtraction.

Ex. To find the square root of $14+6\sqrt{5}$.

Let $\sqrt{x}+\sqrt{y}=\sqrt{14+6\sqrt{5}}$,

then $\sqrt{x}-\sqrt{y}=\sqrt{14-6\sqrt{5}}$,

$\therefore$ multiplying, $x-y=\sqrt{196-180}$,

$=\sqrt{16}=4$.

Also $x+y=14$,

$\therefore x=9, y=5$;

and $\sqrt{x}+\sqrt{y}=3+\sqrt{5}$.

Exercise 55.

(1) Simplify $\sqrt{a^3b}-2\sqrt{ab^3}+\sqrt{abc^2}$; and $\sqrt{12}+2\sqrt{27}-\sqrt{48}$.

(2) Multiply $5\sqrt{2}-7$ by $3\sqrt{2}+1$; and $2\sqrt{ab}+3\sqrt{bc}$ by $3\sqrt{ab}-2\sqrt{bc}$.

(3) Obtain the rationalising factors for $\sqrt{7}+\sqrt{5}$, $2\sqrt{5}-\sqrt{6}$, $a-\sqrt{b}$, $\sqrt[3]{a}+\sqrt[3]{b}$.

(4) *Simplify* $\dfrac{4-\sqrt{2}}{\sqrt{2}+1}$, $\dfrac{6}{5-2\sqrt{6}}$, $\dfrac{a}{\sqrt{b}-\sqrt{c}}$, $\dfrac{2x-\sqrt{xy}}{\sqrt{xy}-2y}$.

Extract the square roots of

(5) $7+4\sqrt{3}$. (6) $20-8\sqrt{6}$. (7) $17+4\sqrt{15}$.

(8) $9-6\sqrt{2}$. (9) $11-6\sqrt{2}$. (10) $2a+2\sqrt{a^2-b^4}$.

(11) $3(a-b)-2\sqrt{2a^2-5ab+2b^2}$. (12) $a^2-2b\sqrt{a^2-b^2}$.

Simplify

(13) $\dfrac{2\sqrt{5}}{\sqrt{5}+1}+\dfrac{2\sqrt{5}}{\sqrt{5}-1}$. (14) $\dfrac{1}{\sqrt{(14+8\sqrt{3})}}$.

(15) $\dfrac{1}{\sqrt{(11+6\sqrt{2})}}-\dfrac{1}{\sqrt{(11-6\sqrt{2})}}+\dfrac{1}{\sqrt{(17-12\sqrt{2})}}$.

N.B.—A root may often be obtained by inspection; for this purpose write the given expression in the form $a+2\sqrt{b}$, and determine what two numbers have their sum $= a$, and their product $= b$.

134.—Equations involving surds may often be solved by arranging the terms so as to have the surd quantity alone on one side, and then raising both sides to such a power that the radical sign may be removed.

Ex. Solve $x+\sqrt{x^2+x}=1$.

Transposing $\quad x-1=-\sqrt{x^2+x}$;

squaring, $\quad x^2-2x+1=x^2+x \; . \; . \; . \; . \; . \; (a)$

$$3x=1,$$

$$x=\frac{1}{3}.$$

Note.—It should be observed that the equation (a) would have also been obtained if the preceding line had been

$$x-1=\sqrt{x^2+x};$$

i.e. if the given equation had been $x-\sqrt{x^2+x}=1$; so that it would appear as if $x=\frac{1}{3}$ were the root of each of the

equations, $\quad x+\sqrt{x^2+x}=1$,

and $\quad x-\sqrt{x^2+x}=1$.

It will be found on trial that it is the root only of the former; the root of the latter indeed, if it could be found, would be clearly an impossible quantity, $\sqrt{x^2+x}$ being obviously greater than x.

Exercise 56.

Solve

(1) $\sqrt{x-5}=2.$ (2) $3-2\sqrt{x^2-1}=2x.$

(3) $2\sqrt{3x+4}-x=4.$ (4) $\sqrt{3x-2}=2(x-4).$

(5) $2x-\sqrt[3]{8x^3+26}+2=0.$

(6) $\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}.$

(7) $\sqrt{2x+1}-\sqrt{x+4}=\frac{1}{3}\sqrt{x-3}.$

(8) $\dfrac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7.$

NEGATIVE AND FRACTIONAL INDICES.

135.—Hitherto the only indices employed to indicate powers have been positive and integral, as in a^3, b^4, x^2, &c. But a meaning may also be attached to indices which are *negative* or *fractional*; and such a meaning that the same rules may be applied to them as are applicable to those which are positive integers.

Negative Indices.—The meaning of a negative index will be suggested by noticing that in a series of descending positive powers of a,

$$a^n, \;.\;.\;.\;.\; a^4,\; a^3,\; a^2,\; a^1,$$

the subtraction of 1 from the index is a division by a. Suppose that the same principle extended farther, to

$$a^0,\; a^{-1},\; a^{-2}, \;.\;.\;.\;.\; a^{-n},$$

then a^0 would equal $\frac{a}{a}$ i.e. 1,

a^{-1} . . . . . . $1 \div a$, i.e. $\frac{1}{a}$,

a^{-2} . . . . . . $\frac{1}{a} \div a$, i.e. $\frac{1}{a^2}$,

and generally

a^{-n} . . . . . . $\frac{1}{a^n}$;

that is, a *negative* power of a letter would be *the reciprocal of the corresponding positive power.* This, then, is the meaning which will be assigned to the power denoted by a negative index; and with this meaning the same rules will be applicable to negative as to positive indices.

(i) $$a^m \times a^{-n} = a^{m-n}.$$

For $$a^m \times a^{-n} = a^m \times \frac{1}{a^n},$$

$$= a^{m-n} \quad \text{by (12: iii) if } m > n;$$

or $$= \frac{1}{a^{n-m}} \quad \text{if } m < n,$$

$$= a^{m-n} \quad \text{by the above definition.}$$

Similarly it may be proved that

$$a^{-m} \times a^{-n} = a^{-m-n};$$
$$a^m \div a^{-n} = a^{m+n};$$
$$a^{-m} \div a^{-n} = a^{n-m}.$$

Therefore the rules for *multiplieation and division* will hold.

(ii) $$(a^{-m})^n = \left(\frac{1}{a^m}\right)^n = \frac{1}{a^{mn}} = a^{-mn}.$$

$$(a^m)^{-n} = \frac{1}{(a^m)^n} = \frac{1}{a^{mn}} = a^{-mn}.$$

$$(a^{-m})^{-n} = \frac{1}{(a^{-m})^n} = \frac{1}{a^{-mn}} = a^{mn}.$$

Hence the rule (56) of *indices in raising to any power* is applicable.

(iii) $$a^{-m} \times b^{-m} = \frac{1}{a^m} \times \frac{1}{b^m} = \frac{1}{(ab)^m} = (ab)^{-m}:$$

i.e. *the product of the same power of two letters is equal to the product of the letters, raised to that power.*

136.—*Fractional Indices.*—The meaning of a fractional index will be suggested by noticing that the division of an index, when the index is exactly divisible, is equivalent to extracting a root; thus a^3 is the square root of a^6; a^4 is the cube root of a^{12}; &c.

If this extended to expressions whose indices are not exactly divisible, $a^{\frac{3}{2}}$ would be the square root of a^3; $a^{\frac{1}{3}}$ would be the cube root of a; $a^{\frac{5}{3}}$ would be $\sqrt[3]{a^5}$; and generally $a^{\frac{m}{n}}$ would be $\sqrt[n]{a^m}$: i.e. in a fractional index *the numerator would indicate a power to which the letter is to be raised, and the denominator a root to be extracted.*

This meaning, then, will be assigned to a fractional index, and the rules which are applicable to integral indices may be proved to be also applicable to these.

(i) *For multiplication and division.*

$$\begin{aligned}
a^{\frac{m}{n}} \times a^{\frac{p}{q}} &= a^{\frac{mq}{nq}} \times a^{\frac{np}{nq}}, \\
&= \sqrt[nq]{a^{mq}} \times \sqrt[nq]{a^{np}}, \\
&= \sqrt[nq]{a^{mq} \times a^{np}} \text{ by (125)}, \\
&= \sqrt[nq]{a^{mq+np}}, \\
&= a^{\frac{mq+np}{nq}} \text{ by the definition}, \\
&= a^{\frac{m}{n}+\frac{p}{q}}.
\end{aligned}$$

Similarly $a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}}$.

(ii) *For raising to any power.*

When the outer index is integral,

$$\begin{aligned}
\left(a^{\frac{m}{n}}\right)^p &= a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots p \text{ factors}, \\
&= a^{\frac{m}{n}+\frac{m}{n}+\ldots} \text{ by (i)}, \\
&= a^{\frac{mp}{n}}.
\end{aligned}$$

Hence also, taking the pth root of each side,

$$a^{\frac{m}{n}} = \sqrt[p]{a^{\frac{mp}{n}}}.$$

When the outer index is fractional,

$$\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}=\sqrt[q]{\left(a^{\frac{m}{n}}\right)^p},$$
$$=\sqrt[q]{a^{\frac{mp}{n}}},$$
$$=a^{\frac{mp}{nq}} \text{ from above.}$$

(iii) Also $a^{\frac{m}{n}}\times b^{\frac{m}{n}}=\sqrt[n]{a^m}\times\sqrt[n]{b^m},$

$$=\sqrt[n]{a^m b^m},$$
$$=\sqrt[n]{(ab)^m},$$
$$=(ab)^{\frac{m}{n}}.$$

The same rules may similarly be shown to hold when the indices are *negative* fractions.

For instance,

$$a^{\frac{m}{n}}\times a^{-\frac{p}{q}}=a^{\frac{m}{n}}\times\frac{1}{a^{\frac{p}{q}}},$$
$$=a^{\frac{m}{n}-\frac{p}{q}}.$$

Exercise 57.

(1) Write with positive indices a^{-2}, $3x^{-1}y^{-3}$, $6x^{-5}y$, $4a^2b^{-3}x^4y^{-5}$, $\dfrac{2a^{-1}x}{3b^{-2}y^{-3}}$.

(2) Express with radical signs $a^{\frac{2}{3}}$, $x^{\frac{3}{4}}$, $a^{\frac{1}{2}}b^{\frac{1}{3}}$, $4x^{\frac{1}{6}}y^{\frac{5}{6}}$, $7mn^{\frac{3}{5}}$, $a^{-\frac{1}{2}}$, $3b^{-\frac{2}{3}}$, $4x^{\frac{1}{3}}y^{-\frac{2}{3}}$.

(3) Express with fractional indices $\sqrt{x^3}$, $x\sqrt[3]{y^2}$, $3a\sqrt[4]{a}$, $\sqrt[6]{a^3b^2}$, $5\sqrt{a^2bc^3x^4}$.

(4) Show that $(a+b)^{-1}+(a-b)^{-1}=2a(a^2-b^2)^{-1}$.

(5) Simplify $x^{\frac{1}{2}}.\ x^{\frac{1}{3}}.\ y.\ x^{\frac{5}{6}}.\ y^{\frac{3}{4}}.\ x^{\frac{1}{3}}.\ y^{\frac{1}{4}}$; and

$$x^2.\ y^{-2}.\ x^{-1}.\ y^3.\ x^{-3}.\ y.$$

(6) Multiply $x^{\frac{1}{2}} - 2x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + 2x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

(7) Write down the squares of $2x^{\frac{1}{2}}$, $3a^{\frac{2}{3}}b^{\frac{1}{4}}$, $4ab^{-1}$, $a^{\frac{1}{2}} - b^{\frac{1}{2}}$, $a + a^{-1}$, $2a^{\frac{1}{2}}b^{\frac{1}{3}} - a^{-\frac{1}{2}}b^{\frac{2}{3}}$.

(8) Divide $a - b$ by $a^{\frac{1}{3}} - a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$.

(9) Simplify

$$(a^{-1} + b^{-1})^2 - (a^{-2} + b^{-2}) - 2a^{-1}b^{-1}(a + b)(a - b)^{-1}.$$

(10) If $a = 4$, $b = 2$, $c = 1$, find the values of $a^{\frac{1}{2}}b$, $3a^{\frac{1}{4}}b^{\frac{1}{2}}c$, $5ab^{-1}$, $2(ab)^{\frac{1}{3}}$, $a^{-\frac{1}{2}}b^{-1}c^{\frac{2}{3}}$, $12a^{-2}b^{-3}$.

(11) Expand $\left(a^{\frac{1}{2}} - b^{\frac{1}{3}}\right)^3$, $\left(2x^{-1} + x\right)^4$, $\left(ab^{-1} - by^{-1}\right)^6$.

(12) Simplify

$$3\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2 - 4\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) + \left(a^{\frac{1}{2}} - 2b^{\frac{1}{2}}\right)^2.$$

(13) Multiply

$$2a^{\frac{5}{6}} - 3a^{\frac{2}{3}}b^{\frac{1}{6}} + a^{\frac{1}{2}}b^{\frac{1}{3}} - 4a^{\frac{1}{3}}b^{\frac{1}{2}} \text{ by } a + 2a^{\frac{5}{6}}b^{\frac{1}{6}} + 3a^{\frac{2}{3}}b^{\frac{1}{3}}.$$

(14) Divide

$$a^2b^{-2} - 21a^{-1}b + 24a^{-3}b^3 - 8a^{-4}b^4 \text{ by } ab^{-1} - 3 + a^{-1}b.$$

(15) Extract the square root of

$$9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2.$$

(16) Extract the cube root of

$$8x^3 + 12x^2 - 30x - 35 + 45x^{-1} + 27x^{-2} - 27x^{-3}.$$

(17) Simplify $5\sqrt{a^{\frac{2}{3}}x} + 3ax\sqrt{a^{-\frac{4}{3}}x^{-1}} - 2\sqrt{\dfrac{a^{\frac{1}{3}}}{a^{-\frac{1}{3}}x^{-1}}}$;

and $\sqrt{a^{\frac{1}{3}}b} \times \sqrt[3]{a^{\frac{1}{2}}b^2} \times 2\sqrt[4]{a^{\frac{2}{3}}x^3} \div 3\sqrt[6]{b^3x^4}$.

(18) Resolve into prime factors with fractional indices $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$; and multiply them together.

(19) Sum the series:—

(i) $1 + \sqrt{2} + 2 + \ldots\ldots$ to 12 terms;
(ii) $a - 2ab^{-1} + 4ab^{-2} - \ldots\ldots$ to 8 terms.

(20) *Simplify* $\sqrt[3]{4a^{-1}b^2c^{\frac{1}{2}}} \times \sqrt[4]{12a^3b^{-\frac{2}{3}}c^2} \div \sqrt[12]{108a^{-8}b^3c^{-4}}$.

Solve the equations :—

(21) $4x^{\frac{1}{2}}-3\left(x^{\frac{1}{2}}+1\right)\left(x^{\frac{1}{2}}-2\right)=x^{\frac{1}{2}}\left(10-3x^{\frac{1}{2}}\right).$

(22) $\left(x^{\frac{2}{3}}-2\right)\left(x^{\frac{4}{3}}-4\right)=x^{\frac{2}{3}}\left(x^{\frac{2}{3}}-1\right)^2-12.$

(23) $x^3-4x^{\frac{3}{2}}=96.$

(24) $x+x^{-1}=2\cdot 9.$

(25) $(x+1+x^{-1})(x-1+x^{-1})=5\frac{1}{4}.$

(26) $81\sqrt[3]{x}+\frac{81}{\sqrt[3]{x}}=52x.$

(27) $2x^{\frac{1}{2}}+3y^{\frac{1}{3}}=12,\quad 3x^{\frac{1}{2}}-2y^{\frac{1}{3}}=5.$

(28) $x^{\frac{1}{2}}+2a^2x^{-\frac{1}{2}}=3a.$

(29) $2\left(x^{\frac{1}{2}}-1\right)^{-1}-2\left(x^{\frac{1}{2}}-4\right)^{-1}=3\left(x^{\frac{1}{2}}-2\right)^{-1}.$

(30) $\frac{x^{-n}}{1+x^{-n}}+\frac{1-x^{-n}}{x^{-n}}=\frac{1}{6}.$

CHAPTER XXV.

THEORY OF QUADRATIC EQUATIONS. RESOLUTION INTO FACTORS. MAXIMA AND MINIMA.

137.—PROP. *If* α *and* β *represent roots of a quadratic equation,* $x^2+px+q=0$, *then* $\alpha+\beta=-p$, *and* $\alpha\beta=q$.

Solving the equation by transposing q and completing the square,

$$x^2+px+\frac{p^2}{4}=\frac{p^2-4q}{4}.$$

$$\therefore \quad x+\frac{p}{2}=\pm\frac{1}{2}\sqrt{p^2-4q},$$

and $$x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$$

Hence, $$\alpha \text{ being } -\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q},$$

$$\text{and } \beta \ \ldots \ -\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q},$$

$$\alpha+\beta=-p,$$

and $$\alpha\beta=\frac{p^2}{4}-\frac{1}{4}(p^2-4q),$$
$$=q.$$

Hence the equation may be written,

$$x^2-(\alpha+\beta)x+\alpha\beta=0,$$

$$\text{or} \quad (x-\alpha)(x-\beta)=0 \quad \ldots\ldots \text{(i)},$$

a form which presents α and β obviously as roots, since when either of them is written for x, one of the factors on the left side $=0$, and the equation is satisfied.

It follows from (i) that a quadratic equation *cannot have more than two roots,* for no value of x except α or β can *make either of the factors* in (i) equal to 0.

138.—The relations proved in the above proposition between the coefficients of an equation and its roots enable any symmetrical expression in a and β to be expressed in terms of p and q.

Ex. To find $a^2+\beta^2$ in terms of p and q.

$$a^2+2a\beta+\beta^2=p^2,$$

and $$2a\beta=2q.$$

$$\therefore \quad a^2+\beta^2=p^2-2q.$$

139.—If the given equation be

$$ax^2+bx+c=0,$$

then $a+\beta$ will $=-\frac{b}{a}$, and $a\beta=\frac{c}{a}$.

140.—An inspection of the roots $a=-\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q}$,

and $\beta=-\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q}$,

of the equation $$x^2+px+q=0,$$
will enable us to determine some characteristics of the roots of an equation without actually solving it:—

(i) As the two roots have the same surd $\sqrt{p^2-4q}$, they will either be *both* possible or *both* impossible;
and if they are possible, they will either be *both* rational or *both* surds:

(ii) that the roots may be *rational*, p^2-4q must be a perfect square,

as in $x^2-5x+6=0$;

(iii) if $p^2=4q$, $\sqrt{p^2-4q}=0$, and the two roots will be *equal*,

as in $x^2-6x+9=0$;

(iv) if $p^2>4q$, $\sqrt{p^2-4q}$ is possible, and the roots will be *possible*,

as in $x^2+3x+1=0$;

(v) if $p^2<4q$, $\sqrt{p^2-4q}$ is impossible, and the roots will be *impossible*,

as in $x^2+3x+4=0$;

(vi) if q (i.e. $a\beta$) be positive, the roots, if real, will have

the *same sign*, which will be opposite to that of p (since $\alpha+\beta=-p$),

as in $x^2-7x+12=0$;

if q be negative, the roots will have *opposite signs;* and by (iv) they will be possible,

as in $x^2-7x-30=0$;

(vii) if p (i.e. $\alpha+\beta$) $=0$, the roots will be *equal*, but of *opposite signs;* in this case they will be possible or impossible according as q is negative or positive,

as in $x^2-4=0$, $x^2+4=0$.

Exercise 58.

If α and β are the roots of the equation $x^2+px+q=0$, find in terms of p and q

(1) $\alpha-\beta$. (2) $\alpha^2-\beta^2$. (3) $\alpha^2\beta+\alpha\beta^2$.
(4) $\alpha^3+\beta^3$. (5) $\alpha^4+\beta^4$. (6) $\alpha^3\beta+\alpha\beta^3$.

(7) Find the sum of the squares, and of the cubes, of the roots of $x^2+6x+4=0$, and of $3x^2-x-1=0$.

Determine without solution the character of the roots of the following equations:—

(8) $x^2+4x-5=0$. (9) $x^2+4x+1=0$.
(10) $x^2-2x+9=0$. (11) $3x^2-4x-4=0$.
(12) $3x^2-8=0$. (13) $x^2+8x+16=0$.

Form, by (137), the equations whose roots are

(14) 1, 2. (15) -3, 7. (16) $\frac{1}{2}$, $\frac{1}{3}$.
(17) -5, $-\frac{1}{2}$. (18) $\cdot 3$, $\cdot 5$. (19) $-\frac{7}{9}$, $\frac{9}{7}$.
(20) $a-2b$, $3a+2b$.

RESOLUTION INTO FACTORS.

141.—Since in (137) $p=-(\alpha+\beta)$, and $q=\alpha\beta$, therefore for *all* values of x (whether they satisfy the equation $x^2+px+q=0$ or not) the expression

$$x^2+px+q=x^2-(\alpha+\beta)x+\alpha\beta,$$
$$=(x-\alpha)(x-\beta).$$

This then supplies a method for resolving a quadratic expression into factors, viz.: *Find* α, β, *the two values of* x *which will make the expression* $=0$, *then* $x-\alpha$ *and* $x-\beta$ *will be the factors required.*

If the expression be given in the form ax^2+bx+c, the factors will be $a(x-\alpha)(x-\beta)$, and it will be found that, if a, b, c are integral, factors of a may be combined with the binomial factors $x-\alpha$, $x-\beta$, so as to make them integral.

Ex. Resolve $12x^2+44x-45$ into factors.

The roots of the equation $\quad 12x^2+44x-45=0,$

$$\text{or} \quad x^2+\frac{44}{12}x-\frac{45}{12}=0,$$

$$\text{are} \qquad \frac{5}{6} \text{ and } -\frac{9}{2};$$

$$\therefore \quad 12x^2+44x-45=12\left(x-\frac{5}{6}\right)\left(x+\frac{9}{2}\right),$$
$$=(6x-5)(2x+9).$$

Exercise 59.

Resolve into factors—

(1) $x^2-4x-221.$ (2) $6x^2-49x+30.$

(3) $14x^2+47x-7.$ (4) $10x^2-11x-18.$

(5) $x^2+ax-6a^2+5ab-b^2.$

(6) $x^2+2xy-3y^2-5x+y+4.$

(7) $x^2-a^2x-x+a-a^3.$

(8) $a^2bx^2+a^3xy-b^3xy-ab^2y^2-bc^2x-ac^2y.$

142.—Prop. *If* $\mathrm{x-a}$ *be a factor of* $\mathrm{x^2+px+q}$ *for all values of* x, *then* $\mathrm{a^2+pa+q}$ *will equal* 0.

Three proofs may be given for this proposition:—

(α) By (141) a must be a root of the equation

$$x^2+px+q=0,$$

and will therefore satisfy it,

$$i.e. \ a^2+pa+q=0.$$

(β) Dividing x^2+px+q by $x-a$,

$$x-a \,/\, x^2+px+q \,\backslash\, x+(a+p)$$
$$\frac{x^2-ax}{(a+p)x+q}$$
$$\frac{(a+p)x-(a^2+pa)}{a^2+pa+q}$$

and as this will be obviously the end of the division, the remainder should be o.

$$\therefore \; a^2+pa+q=0.$$

(γ) Suppose the division of x^2+px+q by $x-a$ to be completed, so that the remainder may not contain x; and let Q represent the quotient and R the remainder; then

$$x^2+px+q=Q(x-a)+R.$$

Now as R does not contain x, it will remain constant, whatever value may be given to x.

Let x be made equal to a, then the above becomes

$$\begin{aligned} a^2+pa+q &= Q\times 0+R, \\ &= R \quad \ldots\ldots\ldots\ldots (1) \end{aligned}$$

But if $x-a$ is a factor of x^2+px+q, R must $=0$,

$$\therefore \; a^2+pa+q=0.$$

N.B.—If $x-a$ is *not* a factor of x^2+px+q, (1) shows that the remainder after the division will be a^2+pa+q.

143.—Prop. Prove (as in 142. γ) that *if*

$$x^n+px^{n-1}+qx^{n-2}+ \;\ldots\; +v$$

be divided by $x-a$, *the remainder will be*

$$a^n+pa^{n-1}+qa^{n-2}+ \;\ldots\; +v.$$

144.—Prop. *If* n *be an odd integer*, x^n+y^n *will be exactly divisible by* $x+y$, *and* x^n-y^n *by* $x-y$.

The division of x^n+y^n by $x+y$ having been carried on until the remainder does not contain x, let Q be the quotient, and R the remainder:

$$\text{then} \quad x^n+y^n=Q(x+y)+R.$$

Now R will contain y but not x, and will therefore remain constant, whatever value may be given to x.

$$\text{Let} \quad x=-y,$$

$$\text{then} \; (-y)^n+y^n=Q(-y+y)+R.$$

But n being odd, $\quad (-y)^n=-y^n$;

$$\therefore \; R=0,$$

which proves the first part of the proposition.

The second part will be proved in the same way, R being found by putting x equal to y.

145.—Prop. *If* n *be an even integer*, x^n-y^n *will be exactly divisible by both* $x+y$ *and* $x-y$; *but* x^n+y^n *by neither*.

For with the same notation as above, we have after the division of x^n-y^n by $x+y$,

$$x^n-y^n = Q(x+y)+R.$$
$$\text{Let } x=-y,$$
$$\text{then } R=(-y)^n-y^n,$$
$$=y^n-y^n \text{ (since } n \text{ is even)},$$
$$=0 \ldots\ldots\ldots\ldots (1)$$

And after the division by $x-y$,

$$x^n-y^n=Q(x-y)+R.$$
$$\text{Let } x=y,$$
$$\text{then } R=y^n-y^n=0 \ldots\ldots\ldots (2)$$

But for the division of x^n+y^n,

$$x^n+y^n=Q(x+y)+R,$$
$$\text{and if } x=-y,$$
$$R=(-y)^n+y^n=2y^n \ldots\ldots (3)$$

Similarly if $\quad x^n+y^n=Q(x-y)+R,$

$$\text{when } x=y,\ R=2y^n \ldots\ldots\ldots (4)$$

$\therefore$ x^n+y^n, when n is even, is divisible by neither $x+y$ nor $x-y$.

The last two propositions will be easily remembered by their simplest cases, viz. when $n=1$ as an odd integer, and when $n=2$ as an even integer.

Exercise 60.

(1) Show without division that $x^5-3x^4+5x^3-24$ is exactly divisible by $x-2$.

(2) What number must be added to x^6+x^5 in order that it may contain $x+3$ as a factor?

(3) Determine the remainder when $x^4-x^3+x^2-x+1$ is divided by $x+6$.

(4) Show that if $x^n+px^{n-1}+\ldots\ldots+v$ is divisible by $x-1$, the sum of the coefficients will equal 0; if by $x+1$, the sums of the coefficients of alternate terms are equal.

(5) Find the quotient when x^n-y^n is divided by $x-y$; and show that $x^n-nxy^{n-1}+(n-1)y^n$ is divisible by $(x-y)^2$.

MAXIMA AND MINIMA.

146.—The solution of a quadratic equation is also useful for the determination of some cases of *maxima* and *minima* values, *i.e.* for finding the greatest and least possible values of quantities subject to certain conditions.

Ex. 1. Find the maximum or minimum value of $4+6x-x^2$ for possible values of x.

$$\begin{aligned} \text{Let} \quad & 4+6x-x^2=m, \\ \text{then } & x^2-6x=4-m, \\ & x^2-6x+9=13-m, \\ \therefore\ & x=3\pm\sqrt{13-m}. \end{aligned}$$

Now as x is possible, $13-m$ cannot be negative, *i.e.* m cannot be greater than 13; so that 13 is the *maximum* value of m; and for that value, $x=3$.

Ex. 2. The difference of two numbers being given $=a$, determine their values in order that the third proportional to the less and greater may be a minimum.

Let x and $x+a$ be the numbers,

then $\dfrac{(x+a)^2}{x}$ is to have its minimum value.

$$\begin{aligned} \text{Let} \quad & \frac{(x+a)^2}{x}=m, \\ & x^2+2ax+a^2=mx, \\ & x^2+(2a-m)x=-a^2 \quad \dots\dots\dots\dots (a) \end{aligned}$$

from which $$x=\frac{m-2a}{2}\pm\frac{1}{2}\sqrt{m^2-4am}.$$

$\therefore$ m^2-4am cannot be negative,
or m cannot be less than $4a$;
in which case $x=a$, and $x+a=2a$.

N.B. x would be possible in the above equation when $m=0$, but the numbers would then be $-a$, 0, which would not be a solution of the problem.

Without actually solving the equation, (140. v) may be applied.

Thus in Ex. (2) we should have from (a),

$$(2a-m)^2 \text{ cannot be } < 4a^2,$$
$$\textit{i.e. } 4am-m^2 \text{ is not } < 0,$$

Exercise 61.

Find the maxima or minima values (determining which) of the following:—

(1) $1+x-x^2$. (2) x^2+3x+4. (3) $(a-x)(x-b)$.

(4) $\dfrac{x}{1+x^2}$. (5) $x+\dfrac{1}{x}$. (6) $\dfrac{x^2}{(x+a)(x-b)}$.

(7) Divide a line a into two parts such that (i) the sum of their squares may be the least possible, (ii) the rectangle contained by the parts may be the greatest possible.

(8) Find the fraction which has the greatest excess over its square.

(9) Describe a right-angled triangle on a given hypothenuse (a) such that its area may be a maximum.

(10) Having given a perimeter ($2s$) and a base (a) for a triangle, find the other sides so that its area may be the greatest possible.

(N.B. x, y, being the other sides, area $=\sqrt{s(s-a)(s-x)(s-y)}$).

CHAPTER XXVI.

ELIMINATION.

147.—When two equations are given, which are satisfied by the same values of one or more letters, a new equation may be formed from them, from which some one of these letters shall be excluded. This letter is then said to be *eliminated.*

The process of elimination has been already employed in the solution of both simple and quadratic equations of more than one unknown; but it may be employed for other purposes also, *e.g.* for finding the *relations* which must exist between the coefficients involved in two equations, in order that these may be simultaneous.

For the elimination of a letter x from two simultaneous simple equations

$$ax+by+c=0 \quad \ldots\ldots\ldots \quad (1)$$
$$a'x+b'y+c'=0 \quad \ldots\ldots\ldots \quad (2)$$

the process given in (52. *First Method*) may be used; viz. multiplying (1) by a', and (2) by a, and subtracting,

$$\text{then } (a'b-ab')y+(a'c-ac')=0.$$

148.—By the extension of this process, as in (53), *two* letters may be eliminated from *three* simultaneous simple equations:—

$$ax+by+cz+d=0 \quad \ldots\ldots \quad (1)$$
$$a'x+b'y+c'z+d'=0 \quad \ldots\ldots \quad (2)$$
$$a''x+b''y+c''z+d''=0 \quad \ldots \quad (3).$$

But it will not be difficult to find in this case multipliers for each of the given equations, such that when the three resulting *equations* are added together *only one of the unknowns will appear.*

Let λ, μ, ν be the three multipliers, then $\lambda(1)+\mu(2)+\nu(3)$ gives

$$(\lambda a+\mu a'+\nu a'')x+(\lambda b+\mu b'+\nu b'')y+(\lambda c+\mu c'+\nu c'')z$$
$$+(\lambda d+\mu d'+\nu d'')=0.$$

Now if y and z are to disappear from this equation, their coefficients must $=0$, so that

$$\left.\begin{aligned}\lambda b+\mu b'+\nu b''&=0\\ \text{and}\quad \lambda c+\mu c'+\nu c''&=0\end{aligned}\right\}.$$

From these $$\frac{\lambda}{\mu}=\frac{b'c''-b''c'}{b''c-bc''}$$

and $$\frac{\lambda}{\nu}=\frac{b'c''-b''c'}{bc'-b'c}.$$

These last results may be written

$$\frac{\lambda}{b'c''-b''c'}=\frac{\mu}{b''c-bc''}=\frac{\nu}{bc'-b'c};$$

they show that the multipliers must be proportional to $b'c''-b''c'$, $b''c-bc''$, and $bc'-b'c$, and may be taken equal to them.

The above expressions will be easily remembered if the arrangement in them of the coefficients b', c', ... be observed. The method by which they were obtained is called that of *Indeterminate Multipliers.*

If the three given equations contain no term which is independent of the unknowns, it will be seen that the resulting equation obtained as above will contain x as a factor in all its terms. The equation may therefore be divided out by x, and the three letters will then all be eliminated.

149.—When a letter is involved to the first degree in one equation, and to a higher degree in the other, it may be eliminated by substitution as in (74. *First Method*).

If a letter is involved to the second or a higher degree in both equations, it may be eliminated as in the following example:—

Let $\quad ax^2+bxy+cy^2+d=0$ . . . . . . (1)

and $\quad a'x^2+b'xy+c'y^2+d'=0$ . . . . . . (2)

be two simultaneous equations, from which x is to be eliminated.

First eliminate x^2;

$$(1) \times a' - (2) \times a \text{ gives}$$

$$(a'b-ab')xy+\{(a'c-ac')y^2+a'd-ad'\}=0 \quad \ldots (3)$$

Next eliminate between (1) and (2) the terms which do not involve x,

$$(1) \times (c'y^2+d') - (2) \times (cy^2+d) \text{ gives}$$

$$\{(ac'-a'c)y^2+ad'-a'd\}x^2+\{bc'-b'c)y^2+bd'-b'd\}xy=0,$$

or, dividing by x,

$$\{(ac'-a'c)y^2+ad'-a'd\}x+\{(bc'-b'c)y^2+bd'-b'd\}y$$
$$=0 \quad \ldots (4)$$

Between (3) and (4), x may then be eliminated, resulting in a biquadratic equation in y.

The above example will show that elimination from general equations soon becomes laborious: but in some particular cases, as in the following examples, means may be devised for effecting it without difficulty.

Exercise 62.

(1) Find the relation which must exist between the coefficients in order that

$$ax^2+bx+c=0,$$
$$\text{and} \quad px^2+qx+r=0,$$

may be satisfied by the same values of x.

(2) Eliminate x and y from

$$\left.\begin{array}{r} x+y=a \\ x^2+y^2=b \\ xy=c \end{array}\right\}$$

(3) From

$$\left.\begin{array}{r} x-y=a \\ x^2-y^2=b \\ xy=c \end{array}\right\}$$

(4) From $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{b^2}-\frac{y^2}{a^2}=0$, $\frac{x^4}{a^4}-\frac{y^4}{b^4}=\frac{h}{k}$.

(5) From $x+y=a$, $x^2+y^2=b^2$, $x^3+y^3=c^3$.

(6) Eliminate x, y, z from $yz=a^2$, $zx=b^2$, $xy=c^2$, and $x^2+y^2+z^2=1$.

(7) From

$$ax+by+cz=0,\ bx+cy+az=0, \text{ and } cx+ay+bz=0.$$

(8) From $x(y+z)=a^2$, $y(z+x)=b^2$, $z(x+y)=c^2$, and $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}$.

(9) From $x^2+y^2=2lxy$, $x^2+z^2=2mxz$, $y^2+z^2=2nyz$.

(10) Eliminate p, q, r from $p^2(q+r)=x^3$, $q^2(p+r)=y^3$, $r^2(p+q)=z^3$, $pqr=xyz$.

CHAPTER XXVII.

HARDER EQUATIONS.

150.—Equations may be solved with respect to an *expression* in the same manner as they have been solved with respect to a letter; and thus the methods which have been employed in quadratics may be applied to the solution of harder equations.

Ex. (1) Solve $(x^2-x)^2-8(x^2-x)+12=0$.

Regard (x^2-x) as the unknown quantity,

$$(x^2-x)^2-8(x^2-x)=-12,$$

and complete the square,

$$(x^2-x)^2-8(x^2-x)+16=4.$$

Extract the square root,

$$x^2-x-4=\pm 2,$$

$$\therefore \quad x^2-x=6 \text{ or } 2.$$

$$x^2-x+\frac{1}{4}=\frac{25}{4} \text{ or } \frac{9}{4},$$

$$x-\frac{1}{2}=\pm\frac{5}{2} \text{ or } \pm\frac{3}{2},$$

$$\text{and } x=\frac{1}{2}\pm\frac{5}{2} \text{ or } \frac{1}{2}\pm\frac{3}{2},$$

$$=3 \text{ or } -2, \text{ or } 2 \text{ or } -1.$$

Ex. (2) Solve $x^2-3x-6\sqrt{x^2-3x-3}+2=0$.

Here by transposing and annexing -3 to x^2-3x,

$$(x^2-3x-3)-6\sqrt{x^2-3x-3}=-2-3,$$

$$=-5,$$

in which we have (x^2-3x-3) and its square root. This may therefore be solved like a quadratic with respect to $\sqrt{x^2-3x-3}$,

$$(x^2-3x-3)-6\sqrt{x^2-3x-3}+9=4,$$

$$\sqrt{x^2-3x-3}-3=\pm 2,$$

$$\sqrt{x^2-3x-3}=5 \text{ or } 1 \quad \ldots\ldots (a)$$

$$x^2-3x-3=25 \text{ or } 1,$$

from which $x=7$ or -4 or 4 or -1.

The same remark applies here as (133, *note*), viz. that in the squaring, new roots may have been introduced; so that until they have been tested it will not be certain that all the values obtained for x will be roots of the given equation.

In this example it will be found that all the values will satisfy the equation.

A little consideration will show that the roots not belonging to the original equation are those which spring from negative quantities on the right-hand side in the line (a): such may therefore be disregarded.

It is sometimes convenient to write another letter for the expression with respect to which the equation is solved; thus the equation above might have been written

$$y^2-6y=-5,$$

y being written for $\sqrt{x^2-3x-3}$.

Thence $\qquad y=5$ or 1,

and $\qquad y^2$ or $x^2-3x-3=25$ or 1,

and so on, as above.

Ex. (3) Solve $\quad x^2+x+\frac{1}{x}+\frac{1}{x^2}=6\frac{3}{4}$.

$$\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=\frac{27}{4}.$$

Now by adding 2 to the $\left(x^2+\frac{1}{x^2}\right)$, we have $\left(x^2+2+\frac{1}{x^2}\right)$,

which is the square of $\left(x+\frac{1}{x}\right)$; and the above becomes

$$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=\frac{35}{4}.$$

Solve with respect to $\left(x+\frac{1}{x}\right)$,

$$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)+\frac{1}{4}=9.$$

$$x+\frac{1}{x}+\frac{1}{2}=\pm 3,$$

$$x+\frac{1}{x}=\frac{5}{2} \text{ or } -\frac{7}{2},$$

$$\therefore \quad x^2 - \frac{5}{2}x = -1,$$

from which $x = 2$ or $\frac{1}{2}$.

Or $$x^2 + \frac{7}{2}x = -1,$$

from which $x = -\frac{1}{4}(7 \pm \sqrt{33})$.

An equation such as the above, which will remain unaltered when for x is written $\frac{1}{x}$, is called a *Reciprocal* equation. Its roots, if it is of an *even* degree, will form pairs which are reciprocals of one another: if of an *odd* degree, as

$$6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0,$$

one root will be $+1$ or -1 (according as the last term is negative or positive) and the rest will form reciprocal pairs.

Ex. (4) Solve $\sqrt{2x+a} + \sqrt{2x-a} = b$.

In this equation it may be noticed that the squares of the two terms on the left side are $2x+a$ and $2x-a$, and the difference of these squares is $2a$.

Now the difference of the squares is divisible by the sum of the terms; hence

$$\frac{2a}{\sqrt{2x+a} + \sqrt{2x-a}} = \frac{2a}{b}$$

becomes $$\sqrt{2x+a} - \sqrt{2x-a} = \frac{2a}{b}.$$

Add to the given equation, then

$$2\sqrt{2x+a} = b + \frac{2a}{b},$$

$$\therefore \quad 8x + 4a = b^2 + 4a + \frac{4a^2}{b^2},$$

and $$x = \frac{b^2}{8} + \frac{a^2}{2b^2}.$$

Ex. (5) Solve $x^4 - x^2y^2 + y^4 = 13 \dots\dots (1)$

$$x^2 - xy + y^2 = 3 \dots\dots (2)$$

Transpose (2).

$$x^2 + y^2 = xy + 3,$$

$\therefore$ squaring, $x^4 + 2x^2y^2 + y^4 = x^2y^2 + 6xy + 9.$

Subtract (1) from this, and we shall obtain

$$2x^2y^2 - 6xy = -4,$$

from which $xy = 1$ or $2 \dots (3)$

If $xy = 1$, $(2)+(3)\times 3$ gives

$$x^2 + 2xy + y^2 = 6,$$
$$\therefore x + y = \pm\sqrt{6},$$

and $(2)-(3)$ gives

$$x^2 - 2xy + y^2 = 2,$$
$$\therefore x - y = \pm\sqrt{2}.$$

Hence $x = \frac{1}{2}(\pm\sqrt{6} \pm \sqrt{2})$, $y = \frac{1}{2}(\pm\sqrt{6} \pm \sqrt{2})$.

If $xy = 2$, the same steps will give

$$x + y = \pm 3,$$
$$x + y = \pm 1,$$

and $x = \pm 2$ or ± 1, $y = \pm 1$ or ± 2.

Other methods of obtaining the roots of equations will suggest themselves after some practice. If one root (either for a letter or an expression) is observable, (71) must be borne in mind.

Exercise 63.

Solve

(1) $(2x^2 - 3x)^2 - 2(2x^2 - 3x) = 15.$

(2) $(ax - b)^2 + 4a(ax - b) = \frac{9a^2}{4}.$

(3) $3(2x^2 - x) - \sqrt{2x^2 - x} = 2.$

(4) $x^2 + \sqrt{x^2 - 7} = 19.$ (5) $x^2 - \sqrt{2x^2 - 5x} = \frac{5}{2}(x + 3).$

(6) $x^2 + x + \frac{6}{x^2 + x} = 7.$

(7) $\sqrt{x^2 - x + 4} = (x + 3)(x - 4) + 4.$

(8) $\sqrt{x + 5} + \sqrt{x - 3} = \sqrt{x^2 + 4x - 16}.$

(9) $x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=1.$

(10) $x^2+3x-\frac{3}{x}+\frac{1}{x^2}=\frac{7}{36}.$

(11) $\sqrt{x+1}+\sqrt{x-1}=5.$ (12) $x+2-4x\sqrt{x+2}=12x^2$

(13) $\sqrt{3x+5}-\sqrt{3x-5}=4.$

(14) $\sqrt{9x^2+21x+1}-\sqrt{9x^2+6x+1}=3x.$

(15) $x^{\frac{4}{3}}-4x^{\frac{2}{3}}+4x^{-\frac{2}{3}}+x^{-\frac{4}{3}}=-\frac{7}{4}.$

(16) $x+y+\sqrt{x+y}=a,\ x-y+\sqrt{x-y}=b.$

(17) $(2x+3y)^2-2(2x+3y)=8,$
$x^2-y^2=21.$

(18) $x^2+y^2+x+y=48,$
$xy=12.$

(19) $(x-y)^2-3(x-y)=10,$
$x^2y^2-3xy=54.$

(20) $bx+ay+c\sqrt{ax+by-c}=c-\frac{c^2}{4},$
$(a-b)(x-y)=c^2.$

CHAPTER XXVIII.

INEQUALITIES.

151.—Expressions involving any given letter (or, as they are termed, *Functions* of that letter) will have their values changed when different values are assigned to that letter; and of two such expressions, one may be for some values of the letter larger, for others smaller, than the other.

Thus $1+a+a^2$ will clearly be greater than $1-a+a^2$ for all positive values of a, but less for negative values; so a^2 is $>$ or $<$ than a according as a is $>$ or $<$ than 1.

But one expression is sometimes so related to another that whatever values may be given to the letters, it can never become greater than that other.

Thus $2a$ cannot be $> a^2+1$, whatever value be given to a.

For determining whether such a relation holds between two expressions, the following is a fundamental proposition:—

PROP. *If* a *and* b *are unequal,* $a^2+b^2>2ab$.

For $(a-b)^2$ must be positive whatever a and b are; *i.e.*

$$(a-b)^2>0,$$
$$\text{or } a^2-2ab+b^2>0,$$
$$\therefore \quad a^2+b^2>2ab.$$

The principles upon which the solution of equations depends (22) are applicable to inequalities, except that if *each side* of an inequality have its sign changed, the *inequality will* be reversed; thus if $a>b$, $-a$ will be $<-b$.

Ex. (1) Which is the greater, a^3+b^3 or a^2b+ab^2, a and b being positive?

$$a^3+b^3 \text{ is } > \text{ or } < a^2b+ab^2,$$

according as (dividing each side by $a+b$)

$$a^2-ab+b^2 > \text{ or } < ab,$$

according as $\quad a^2+b^2 > \text{ or } < 2ab.$

But $\quad a^2+b^2 \text{ is } > 2ab,$

$$\therefore \quad a^3+b^3 \text{ is } > a^2b+ab^2.$$

Ex. (2) Prove that $a^2+b^2+c^2$ is $> ab+ac+bc$.

$$a^2+b^2 > 2ab,$$
$$a^2+c^2 > 2ac,$$
$$b^2+c^2 > 2bc,$$
$$\therefore \quad 2a^2+2b^2+2c^2 > 2ab+2ac+2bc,$$
$$\text{or} \quad a^2+b^2+c^2 > ab+ac+bc.$$

Exercise 64.

Show that (the letters being unequal and positive)

(1) $a^2+3b^2 > 2b(a+b)$. (2) $\frac{x}{y}+\frac{y}{x} > 2$.

(3) $a^3b+ab^3 > 2a^2b^2$.

(4) $(a^2+b^2)(a^4+b^4) > (a^3+b^3)^2$.

(5) $a^2b+a^2c+ab^2+b^2c+ac^2+bc^2 > 6abc$.

(6) $(1+a^2+a^4) > \frac{3}{2}(a+a^3)$.

(7) Which is the greater, $(a^2+b^2)(c^2+d^2)$ or $(ac+bd)^2$?

(8) Which is the greater, m^2+m or m^3+1?

(9) Which is the greater, a^4-b^4 or $4a^3(a-b)$? $a > b$.

(10) Which is the greater, $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}$ or $\sqrt{a}+\sqrt{b}$?

CHAPTER XXIX.

BINOMIAL THEOREM WHEN THE INDEX IS FRACTIONAL OR NEGATIVE.

152.—In Chapter XXII. it was proved that when a binomial is raised to any proposed power, for which the index is a positive integer, its expansion proceeds by certain laws, giving rise to one general formula. The same formula may be shown to hold also when the index is fractional or negative.

PROP. *If the two expressions*

$$1+mx+\frac{m(m-1)}{1 \cdot 2}x^2+ \dots .$$

and
$$1+nx+\frac{n(n-1)}{1 \cdot 2}x^2+ \dots .$$

be multiplied together, their product will be a series of the same form, viz.

$$1+(m+n)x+\frac{(m+n)(m+n-1)}{1 \cdot 2}x^2+ \dots .$$

For when any two expressions in x, as

$$1+ax+bx^2+ \dots .$$
$$\text{and} \quad 1+a'x+b'x^2+ \dots .$$

are multiplied together, their product will be an expression in ascending powers of x, as

$$1+Ax+Bx^2+ \dots .$$

in which the coefficients A, B, . . . will be functions of a, b, a', b' . . . .; *i.e.* will be made up of these letters in particular ways. And the way in which a, b, a', b' . . . . enter into these functions will clearly be the same, whatever may be the *values* assigned to the letters.

So in the product of

$$1+mx+\frac{m(m-1)}{1\,.\,2}x^2+\ldots.$$

$$\text{and } 1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\ \ldots.$$

the coefficients of x, x^2, ... will be functions of m and n, whose *form* will remain the same for all values of m and n.

Now we can ascertain the form of the coefficients when m and n are positive integers, for in that case

$$1+mx+\frac{m(m-1)}{1\,.\,2}x^2+\ \ldots \text{ is } (1+x)^m \ \ldots \ (116)$$

$$\text{and } 1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\ \ldots \text{ is } (1+x)^n;$$

$$\therefore \text{ their product is } (1+x)^{m+n},$$

which when expanded becomes

$$1+(m+n)x+\frac{(m+n)(m+n-1)}{1\,.\,2}x^2+\ \ldots.$$

These forms therefore will also hold when m and n are fractional or negative.

153.—If the above expressions are represented by $f(m)$ and $f(n)$, their product, since it is formed with $m+n$ in precisely the same way that they are formed with m and n, will be represented by $f(m+n)$; and the proposition will be

$$f(m)\times f(n)=f(m+n).$$

$$\text{Hence also } f(m)\times f(n)\times f(p)=f(m+n)\times f(p),$$
$$=f(m+n+p),$$

and so on for any number of such factors.

154.—Prop. *To deduce the Binomial Theorem for fractional or negative indices.*

(i) *Fractional.* $f(m)$ meaning the same as above, we have for *all* values of $m, n, p, \ldots$.

$f(m) \cdot f(n) \cdot f(p) \ldots s$ factors $= f(m+n+p+ \ldots$ terms).

If $n, p, \ldots$ be each made $= m$, this becomes

$$\{f(m)\}^s = f(ms)$$

where s is any positive integer.

Now if ms be any positive integer r, so that m becomes the positive fraction $\frac{r}{s}$, the above (being true for *all* values of m) becomes

$$\left\{f\left(\frac{r}{s}\right)\right\}^s = f(r)$$

$$= (1+x)^r \text{ since } r \text{ is integral;}$$

$$\therefore (1+x)^{\frac{r}{s}} = f\left(\frac{r}{s}\right),$$

$$= 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1 \cdot 2}x^2 + \ldots$$

i.e. the expansion of $(1+x)^{\frac{r}{s}}$, in which the index is a positive fraction, is an expression of the same form as when the index is a positive integer.

(ii) *Negative.* Again, $f(m) \cdot f(n) = f(m+n)$ being true for all values of m and n, will hold when $n = -m$.

Hence $f(m) \cdot f(-m) = f(0)$, which equals 1.

$$\therefore \quad f(-m) = \frac{1}{f(m)},$$

$$= \frac{1}{(1+x)^m} \text{ from above;}$$

i.e. $(1+x)^{-m} = f(-m)$.

$\therefore$ when the index is negative (whether integral or fractional) the formula of expansion is still true.

NOTE. The above is known as Euler's proof.

Ex. (1).

$$(1+x)^{\frac{2}{3}} = 1 + \frac{2}{3}x + \frac{\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)}{1.2}x^2 + \frac{\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1.2.3}x^3 + \ldots$$

$$= 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 - \ldots.$$

Ex. (2).

$$\frac{1}{\sqrt{(4a^2-3ax)^3}} = (4a^2-3ax)^{-\frac{3}{2}}.$$

Here $4a^2 - 3ax = 4a^2\left(1 - \frac{3x}{4a}\right)$.

$$\therefore (4a^2-3ax)^{-\frac{3}{2}} = (4a^2)^{-\frac{3}{2}}\left\{1 - \frac{3x}{4a}\right\}^{-\frac{3}{2}},$$

$$= \frac{1}{8a^3}\left\{1 - \left(-\frac{3}{2}\right)\frac{3x}{4a} + \frac{-\frac{3}{2}\cdot-\frac{5}{2}}{1.2}\left(\frac{3x}{4a}\right)^2 - \frac{-\frac{3}{2}\cdot-\frac{5}{2}\cdot-\frac{7}{2}}{1.2.3}\left(\frac{3x}{4a}\right)^3 + \ldots\right.$$

$$= \frac{1}{8a^3}\left\{1 + \frac{9x}{8a} + \frac{135x^2}{128a^2} + \frac{945x^3}{1024a^3} + \ldots\right.$$

A root of a number may sometimes be readily found by means of an expansion.

Ex. (3) Find the cube root of 513 to six decimal places.

$$513 = 512\left(1 + \frac{1}{512}\right) = 8^3\left(1 + \frac{1}{512}\right).$$

$$\therefore \sqrt[3]{513} = 8\left(1 + \frac{1}{512}\right)^{\frac{1}{3}},$$

$$= 8\left\{1 + \frac{1}{3}\cdot\frac{1}{512} - \frac{1}{9}\left(\frac{1}{512}\right)^2 + \ldots\right\}$$

$$= 8\{1 + .00065104 1 - .000000423\}$$

$$= 8.005204.$$

Exercise 65.

Expand to four terms

(1) $(1+x)^{\frac{1}{2}}$. (2) $(a+x)^{\frac{3}{4}}$. (3) $(1-x)^{-4}$.

(4) $(a^2-x^2)^{\frac{5}{3}}$. (5) $(x^2+xy)^{-\frac{3}{2}}$. (6) $(2x-3y)^{-\frac{1}{4}}$.

(7) $\sqrt[5]{1-5x}$. (8) $\dfrac{1}{\sqrt[4]{a^2-2ax}}$. (9) $\sqrt[6]{\dfrac{1}{(1-3y)^5}}$

(10) $(1+x+x^2)^{\frac{2}{3}}$.

(Expand first $\{1+(x+x^2)\}^{\frac{2}{3}}$ in terms of $(x+x^2)$).

(11) $\{1-x+x^2\}^{\frac{3}{2}}$.

(12) Obtain the rth term of $(a+x)^{\frac{1}{2}}$. (*See Art.* 116.)

(13) Obtain the rth term of $(a-x)^{-1}$.

(14) Find to five decimal places $\sqrt{65}$ and $\sqrt[3]{1\frac{1}{30}}$.

(15) Find $\sqrt[7]{129}$ to six decimal places.

(16) Expand $(1-2x+3x^2)^{-\frac{2}{5}}$ to five terms.

(17) Find the coefficient of x^4 in the expansion of

$$\frac{(1+2x)^9}{(1+3x)^3}; \text{ and of } x^r \text{ in } \frac{1+x}{(1-x)^2}.$$

(18) Find $\dfrac{1}{\sqrt[3]{999}}$ to six decimal places.

(19) By means of the expansion in (1) show that the limit of the series

$$1+\frac{1}{2}-\frac{1}{2 \cdot 2^2}+\frac{1 \cdot 3}{2 \cdot 3 \cdot 2^3}-\frac{1 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 2^4}+ \ldots \text{ is } \sqrt{2};$$

and that of the series

$$1+\frac{1}{2 \cdot 2}+\frac{1 \cdot 3}{2 \cdot 3 \cdot 2^2}+\frac{1 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 2^3} \ldots \text{ is } 2.$$

(20) Find the coefficient of x^r in the product of $(1-x)^{-n}$ and $(1-x)^{-1}$; and by comparing this with that of x^r in $(1-x)^{-(n+1)}$, find the sum of the coefficients of the first $r+1$ *terms of* $(1-x)^{-n}$.

CHAPTER XXX.

INDETERMINATE COEFFICIENTS.

155.—The theorem known as the principle of *Indeterminate Coefficients* is this:—

If the two series $a+bx+cx^2+dx^3+\ \ldots\ldots$
and $a'+b'x+c'x^2+d'x^3+\ \ldots\ldots$
be equal for all values of x, *then*

$$a=a',\ b=b',\ c=c',\ \&c.$$

The proof depends upon the preliminary proposition,

If the coefficients $\alpha, \beta, \gamma\ \ldots$ *are finite, a series*

$$\alpha x+\beta x^2+\gamma x^3+\ \ldots\ldots$$

may by taking x *small enough be made less than any quantity that may be assigned.*

For if k be any assigned quantity, and κ represent the greatest of the coefficients $\alpha, \beta, \ldots$

$$\alpha x+\beta x^2+\gamma x^3+\ \ldots \text{ is } < \kappa x+\kappa x^2+\kappa x^3+\ \ldots$$

$$i.e. < \frac{\kappa x}{1-x} \text{ if } x \text{ be taken } < 1.$$

If therefore $\frac{\kappa x}{1-x}$ be taken $< k$,

$$i.e. \text{ if } x \text{ be } < \frac{k}{k+\kappa},$$

$$\alpha x+\beta x^2+\gamma x^3+\ \ldots \text{ will be } < k.$$

To prove then the above theorem:—

Since $a+bx+cx^2+\ \ldots\ = a'+b'x+c'x^2+\ \ldots$ for all values of x, we have, by putting $x = 0$,

$$a = a'.$$

Hence $bx+cx^2+\ \ldots\ = b'x+c'x^2+\ \ldots$

$$\therefore\ (b-b')x = (c'-c)x^2+(d'-d)x^3+\ \ldots$$

$$\therefore\ b-b' = (c'-c)x+(d'-d)x^2+\ \ldots$$

Now by giving x a sufficiently small value, the right side may be made less than any quantity that may be assigned, and so may be made indefinitely small,

$\therefore$ the difference between b and b' is indefinitely small;

$$i.e. \quad b = b'.$$

Similarly $c = c'$, $d = d'$, &c.

The application of the theorem will be seen by the following examples:—

Ex. (1) Expand $\dfrac{2+3x}{1-x+x^2}$ in ascending powers of x.

Let $\dfrac{2+3x}{1-x+x^2} = A+Bx+Cx^2+Dx^3+ \ldots.$

then
$$\begin{aligned} 2+3x &= A+Bx+Cx^2+Dx^3+ \ldots. \\ &\qquad\; -Ax-Bx^2-Cx^3+ \ldots. \\ &\qquad\qquad\; +Ax^2+Bx^3+ \ldots. \\ &= A+(B-A)x+(C-B+A)x^2+(D-C+B)x^3+ \ldots. \end{aligned}$$

$\therefore$ by the theorem,

$$A = 2,\ B-A = 3,\ C-B+A = 0,\ D-C+B = 0.$$

$\therefore$ $B=A+3=5$, $C=B-A=3$, $D=C-B= -2$, &c.

$\therefore$ $\dfrac{2+3x}{1-x+x^2} = 2+5x+3x^2-2x^3- \ldots$ &c.

Ex. (2) Find the fraction in the form $\dfrac{a+bx}{p+qx+rx^2}$ which will produce by its expansion the series

$$1-5x+6x^2+8x^3-40x^4+ \text{\&c.}$$

As the first term of the series is 1, we may assume $\dfrac{1+Bx}{1+Cx+Dx^2}$ to be the fraction required; then (multiplying out by the denominator)

$$\begin{aligned} 1+Bx = 1 &- 5x + 6x^2 + 8x^3 - \ldots \\ &+ Cx - 5Cx^2 + 6Cx^3+ \ldots \\ &\qquad\; + Dx^2 - 5Dx^3+ \ldots \quad \ldots. \ (a) \end{aligned}$$

$\therefore$ $B = -5+C$, $0 = 6-5C+D$, $0 = 8+6C-5D$,
from which $B = -3$, $C=2$, $D=4$.

$\therefore$ the fraction is $\dfrac{1-3x}{1+2x+4x^2}$.

Note.—An examination of the product (a) will show

at if u_r, u_{r+1}, u_{r+2} be any three consecutive terms of the ven series,

$$u_{r+2} + Cx . u_{r+1} + Dx^2 . u_r \text{ will } = 0.$$

A series, any given number of whose consecutive terms e connected in this manner, is said to be *recurring*, and e expression $1 + Cx + Dx^2$ is called its *scale of relation.*

Ex. (3) If $y = ax + bx^2 + cx^3 + \ldots$ find x in terms of y.

$$\begin{aligned} \text{Let } x &= Ay + By^2 + Cy^3 + \ldots \\ &= Aax + Abx^2 + Acx^3 + \ldots \\ &\qquad + Ba^2x^2 + 2Babx^3 + \ldots \\ &\qquad\qquad + Ca^3x^3 + \ldots \end{aligned}$$

$$\therefore Aa = 1, \ Ab + Ba^2 = 0, \ Ac + 2Bab + Ca^3 = 0,$$

from which $A = \frac{1}{a}$, $B = -\frac{b}{a^3}$, $C = \frac{2b^2 - ac}{a^5}$.

$$\therefore x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2b^2 - ac}{a^5} y^3 - \&c.$$

Note.—The above is an example of *Reversion of Series.* t is evident *à priori* that since when $x = 0$ in the given eries, $y = 0$, there will be in the required series no term lear of y: but if the given series should be in the form $+ bx + cx^2 + \ldots .$, then it would be necessary to ke $y - a = z$, so that

$$z = bx + cx^2 + \ldots .$$

nd expand x in terms of z.

Ex. (4) Resolve $\frac{7x + 1}{(x + 4)(x - 5)}$ into its partial fractions, .e. into the simplest fractions which will together be qual to it.

The denominators will be $x + 4$, $x - 5$.

$$\text{Let } \frac{7x+1}{(x+4)(x-5)} = \frac{A}{x+4} + \frac{B}{x-5},$$

$$\text{then } \quad 7x + 1 = A(x-5) + B(x+4),$$

$$\therefore A + B = 7, \ -5A + 4B = 1.$$

$$\text{Whence } \quad A = 3, \ B = 4,$$

$$\therefore \frac{7x+1}{(x+4)(x-5)} = \frac{3}{x+4} + \frac{4}{x-5}.$$

Ex. (5) Resolve $\dfrac{4x^3-x^2-3x-2}{x^2(x+1)^2}$ into partial fractions.

Here the denominators may be x, x^2, $x+1$, $(x+1)^2$.

$$\text{Let } \frac{4x^3-x^2-3x-2}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

$$\begin{aligned}\text{Then } 4x^3-x^2-3x-2 &= Ax(x+1)^2+B(x+1)^2 \\ &\qquad +Cx^2(x+1)+Dx^2 \\ &= (A+C)x^3+(2A+B+C+D)x^2 \\ &\qquad +(A+2B)x+B\end{aligned}$$

$$\begin{aligned}\therefore\quad A+C &= 4 \\ 2A+B+C+D &= -1 \\ A+2B &= -3 \\ B &= -2\end{aligned}$$

whence $A=1$, $B=-2$, $C=3$, $D=-4$;

$$\text{and the fraction } = \frac{1}{x} - \frac{2}{x^2} + \frac{3}{x+1} - \frac{4}{(x-1)^2}.$$

Ex. (6) Resolve $\dfrac{3}{x^3-1}$ into partial fractions.

The factors of x^3-1 are $x-1$ and x^2+x+1, the latter of which cannot be resolved into simpler factors; for this denominator the numerator may be of the form $Bx+C$.

$$\text{Let } \frac{3}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1},$$

$$\therefore\quad 3 = A(x^2+x+1) + (Bx+C)(x-1);$$

whence as before $A=1$, $B=-1$, $C=-2$,

$$\text{and } \frac{3}{x^3-1} = \frac{1}{x-1} - \frac{x+2}{x^2+x+1}.$$

Note.—In all the above examples the numerator is of lower dimensions than the denominator; when this is not the case, it is best to express the fraction as a mixed number, and then proceed as above.

156.—Resolution into partial fractions affords a simple method of *obtaining any required term of an expansion* such as that of Ex. (1), when the denominator can be *resolved* into factors of the first degree.

Ex. (7) To find the terms involving x^6 and x^r in the expansion of $\frac{3+8x}{(1-2x)(3+x)}$.

$$\frac{3+8x}{(1-2x)(3+x)} = \frac{2}{1-2x} - \frac{3}{3+x}$$

$$= \frac{2}{1-2x} - \frac{1}{1+\frac{x}{3}}$$

$$= 2\left\{1+2x+(2x)^2+\ldots(2x)^6+\ldots\ldots(2x)^r+.\right\}$$

$$- \left\{1-\frac{x}{3}+\left(\frac{x}{3}\right)^2-\ldots+\left(\frac{x}{3}\right)^6-\ldots+\left(-\frac{x}{3}\right)^r+.\right\}$$

$\therefore$ coeff. of $x^6 = 2\times 2^6-\frac{1}{3^6} = 127\,\frac{728}{729}$.

Coeff. of $x^r = 2\times 2^r-\frac{1}{3^r}$ if r is even,

or $2\times 2^r+\frac{1}{3^r}$ if r is odd.

These last two may be expressed in one formula:

$$2^{r+1} - \frac{(-1)^r}{3^r}.$$

Exercise 66.

(1) Expand to four terms in ascending powers of x,

$$\frac{1}{1-2x+3x^2},\quad \frac{5-2x}{1+3x-x^2},\quad \frac{3-2x}{4-3x}.$$

(2) If $y=x+2x^2+3x^3+\ .\ .\ .$ find x in terms of y: also if $y=x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ .\ .\ .\ .$

(3) Find the fractions of the form $\frac{p+qx}{a+bx+cx^2}$ whose expansions produce the series

$$1+3x+\ 2x^2-\ x^3-\ .\ .\ .\ .$$
$$3+2x+\ 3x^2+\ 7x^3+\ .\ .\ .\ .$$
$$\frac{3}{4}-\frac{3x}{16}+\frac{63x^2}{64}-\frac{123x^3}{256}+\ .\ .\ .\ .$$

Resolve into partial fractions:

(4) $\dfrac{3x-7}{(x-2)(x-3)}$, $\dfrac{6}{(x+3)(x+4)}$, $\dfrac{5x-1}{(2x-1)(x-5)}$, $\dfrac{x-2}{x^2-3x-10}$.

(5) $\dfrac{x^2-x-3}{x(x^2-4)}$, $\dfrac{3x^2-4}{x^2(x+5)}$, $\dfrac{7x^2-x}{(x-1)^2(x+2)}$, $\dfrac{5}{x(x-1)(2x-1)^2}$, $\dfrac{2x^2-7x+1}{x^3+1}$.

CHAPTER XXXI.

LOGARITHMS.

(See *Arithmetic*, chap. xvi.)

157.—Any number a being taken as base, the index of the power to which a must be raised in order to equal another number n is called the *Logarithm* of n to the base a.

If $n = a^x$, x is log n to base a; or, as it is written, $x = \log_a n$.

$$\text{Thus since } 9 = 3^2,\ 2 = \log_3 9;$$
$$\text{since } 10000 = 10^4,\ 4 = \log_{10} 10000.$$

The number taken as base will be always considered positive and > 1.

158.—PROP. *The logarithm of the product of two numbers to any base is equal to the sum of their logarithms.*

For let m and n be two numbers,
x and y their logarithms to base a.

$$\begin{aligned} \text{Then}\quad m &= a^x,\ n = a^y, \\ \therefore\quad mn &= a^x . a^y = a^{x+y}. \\ \therefore\ \log_a mn &= x+y \\ &= \log_a m + \log_a n. \end{aligned}$$

159.—PROP. *The logarithm of the quotient of two numbers is equal to the difference of their logarithms.*

For with the same notation as above,

$$\begin{aligned} \frac{m}{n} &= a^x \div a^y = a^{x-y} \\ \therefore\ \log_a \frac{m}{n} &= x-y \\ &= \log_a m - \log_a n. \end{aligned}$$

Hence by supposing $n=m$, $\log_a 1 = 0$;

if $n > m$, $\log_a \frac{m}{n}$ is negative,

i. e. the logarithm of a number less than unity is negative.

160.—PROP. *The logarithm of any power of a number is equal to the logarithm of the number, multiplied by the index of the power.*

$$\begin{aligned} \text{For if} \quad x &= \log_a n, \; n = a^x, \\ \therefore \quad n^m &= a^{mx} \\ \therefore \log_a (n^m) &= mx = m \log_a n. \end{aligned}$$

Since m may in the above proof be considered fractional $\left(=\frac{1}{r}\right)$, we have also

$$\log_a \left(n^{\frac{1}{r}}\right) \text{ or } \log_a \sqrt[r]{n} = \frac{1}{r} \log_a n,$$

i. e. *the logarithm of a root is equal to the logarithm of the number, divided by the index of the root.*

161.—PROP. $\operatorname{Log}_a m = \operatorname{Log}_b m \times \operatorname{Log}_a b.$

$$\begin{aligned} \text{For let } \log_b m &= y, \text{ and } \log_a b = z, \\ \text{then} \quad m &= b^y, \; b = a^z, \\ \therefore \quad m &= (a^z)^y = a^{yz}, \\ \therefore \log_a m &= yz = \log_b m \times \log_a b. \end{aligned}$$

$$\text{Hence } \operatorname{Log}_b m = \frac{\operatorname{Log}_a m}{\operatorname{Log}_a b},$$

so that if the logarithms of numbers to any base a are given, their logarithms to another base b may be obtained by dividing by $\log_a b$.

If $m = a$, the above formula becomes

$$\log_a a \text{ or } 1 = \log_b a \times \log_a b$$

$$\therefore \log_b a = \frac{1}{\log_a b}.$$

Exercise 67

(1) Write down the logarithms to base 9, of 9, 81, 729, 3, 27, 2187, $\frac{1}{3}$, $\frac{1}{81}$, $\frac{1}{243}$, $\frac{1}{\sqrt[4]{3}}$.

(2) Write down the numbers whose logarithms to base 16 are 2, 1, ·5, 2·5, −1, −·5, −·25, ·0625.

(3) If $\log_a x = b$, and $\log_a y = c$, find the logarithms of x^2y, x^3y^4, $x^4y^{\frac{1}{3}}$, $x^{-2}y^3$, $x^py^{-\frac{1}{q}}$.

(4) Having given $\log 2 = \cdot 301$, and $\log 3 = \cdot 477$, find the logarithms of 6, 18, 72, $\frac{1}{12}$, ·25, ·041$\dot{6}$.

(5) ·2, ·3, ·4 being the logarithms of three numbers to base a, what will be their logarithms to base a^2?

(6) If a, b, c are in geometrical progression, show that $\log_n a$, $\log_n b$, $\log_n c$ are in arithmetical, and $\log_a n$, $\log_b n$, $\log_c n$ in harmonical, progression.

(7) Having given $\log_{10} 3 = \cdot 477$, what power is 10 of 3; and 1000 of 9?

(8) Prove that $\log_a b \times \log_b c \times \log_c d \times \ldots \log_h k = \log_a k$.

(9) Given $\log_{10} 2 = \cdot 301$, and $\log_{10} 3 = \cdot 477$, find $\log_2 3$ and $\log_8 9$: and if in another scale $\log 2 = \cdot 693$, find $\log 3$ in that scale.

(10) Given $\log 5 = \cdot 699$ and $\log 7 = \cdot 845$, which is the greater, 5^{12} or 7^{10}? and between what two powers of 5 does the 12th power of 7 lie?

162.—The construction of a table of logarithms is effected by obtaining an expansion for $\log(1+x)$ in terms of x; and *this again depends upon the Exponential Theorem, i. e. the expansion of* a^x in terms of x, as follows.

163.—PROP. *To expand* a^x *in a series of ascending powers of* x.

By the Binomial Theorem,

$$a^x = \{1+(a-1)\}^x = 1+x(a-1)+\frac{x(x-1)}{1.2}(a-1)^2$$
$$+\frac{x(x-1)(x-2)}{1.2.3}(a-1)^3+\ldots$$

Now if each coefficient in this series were expanded, it would consist of powers of x, and the whole series might be then rearranged in ascending powers of x.

The coefficient of x would be

$$(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+ \text{\&c.}$$

which may be denoted by A.

And if B, C, ... be employed to denote the coefficients of x^2, x^3, ..., the first series may be written

$$a^x = 1+Ax+Bx^2+Cx^3+\ldots\ldots$$

in which A has been found, but B, C, ... are still to be determined.

They may be found in terms of A: for, since the above is true for all values of x, we have

$$a^{x+y} = 1+A(x+y)+B(x+y)^2+C(x+y)^3+\ldots$$

But $a^{x+y} = a^y.a^x = a^y\{1+Ax+Bx^2+Cx^3+\ldots\}$; and these two series will be identically equal. Hence by (155) the coefficients of x in the two series will be equal, viz.

$$\begin{aligned} A+2By+3Cy^2+\ldots\ldots &= Aa^y \\ &= A(1+Ay+By^2+Cy^3\ldots) \end{aligned}$$

But these expressions are equal for *all* values of y, therefore the coefficients of y, y^2, y^3, ... in one are equal to those in the other; so that

$$2B = A^2,\quad 3C = AB,\quad \text{\&c.}$$

$$\therefore\ B = \frac{A^2}{1.2},\quad C = \frac{AB}{3} = \frac{A^3}{1.2.3}, \text{\&c.}$$

and therefore

$$a^x = 1+Ax+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\ldots$$

A being, as found above, $(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-$ &c.

Now if $Ax = 1$, or $x = \frac{1}{A}$, the last series becomes

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} +$$

a series whose value can be easily found to be $2 \cdot 7182818 \ldots$ and which is usually denoted by e.

Hence $a^{\frac{1}{A}} = e$, $\therefore a = e^A$, and $A = \log_e a$.

$$\therefore a^x = 1 + (\log_e a)x + (\log_e a)^2 \frac{x^2}{1.2} + (\log_e a)^3 \frac{x^3}{1.2.3} + \ldots$$

N.B.—If $a = e$, then $\log_e a = 1$, and

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1.2.3} + \ldots .$$

164.—PROP. *To expand* $\log_e (1+x)$ *in terms of* x.

In the above proof we have

$$\log_e a = A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \ldots .$$

If then for a we write $1+x$,

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots .$$

165.—The above series, though obtained for all values of x, would be of no practical use for values of x greater than unity, and for smaller values (unless very small) would require a large number of terms to be used. It is necessary, therefore, to obtain from it other formulæ, in the following way:—

Since $\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots .$

$\therefore$ by writing $-x$ for x,

$$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots .$$

Subtracting, $\log_e (1+x) - \log_e (1-x)$ *or*

$$\log_e \left(\frac{1+x}{1-x}\right) = 2 \left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots .\right\}$$

Let $\frac{1+x}{1-x}$ be written $\frac{m}{n}$, so that $x = \frac{m-n}{m+n}$,

then $\log_e \frac{m}{n} = 2 \left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 \ldots\right\}$

$$\text{If } n = 1, \log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \ldots\right\}$$

a series from which the logarithms of the small numbers 2, 3, may be obtained.

$$\text{Again, if } m = n+1, \text{ then } \frac{m-n}{m+n} = \frac{1}{2n+1},$$

$$\text{and } \log_e \frac{n+1}{n} \text{ or } \log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3}\frac{1}{(2n+1)^3} + \ldots\right\}$$

$$\therefore \log_e(n+1) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\frac{1}{(2n+1)^3} + \ldots\right\}$$

by which the logarithm of any number may be readily obtained from that of the preceding number.

166.—Logarithms to the base e are those which are obtained most simply, and are known as *natural*; or *Napierian*, from their having been invented by Napier of Merchiston, and published by him in 1614.

The common system, which is much more convenient for practical use, is known as *Briggs's*, or the *decimal*, system, and is calculated to the base 10. It may be obtained from the Napierian by means of (161), for if N be any number,

$$\text{Log}_{10} N = \frac{\log_e N}{\log_e 10}.$$

The multiplier $\frac{1}{\log_e 10}$, by means of which every natural logarithm may be reduced to the corresponding decimal logarithm, is called the *modulus* of the common system, and $= \cdot 4342944819$.

167.—The last series in (165) may be readily adapted to the common system, for if μ represent the modulus,

$$\log_{10}(n+1) - \log_{10} n = \mu \log_e (n+1) - \mu \log_e n$$

$$= 2\mu\left\{\frac{1}{2n+1} + \frac{1}{3}\frac{1}{(2n+1)^3} + \ldots\right\}$$

and by means of this a logarithm in the common system also may be immediately obtained from the preceding one.

168.—In obtaining the logarithm of a number which consists of more digits than are given in the tables, it is assumed (*Arithmetic*, Art. 100) that if a large number be slightly increased, the increase produced in the logarithm will be proportional to that given to the number.

To prove this, we have now

$$\begin{aligned} \log_{10}(n+d)-\log_{10} n &= \log_{10}\left(1+\frac{d}{n}\right) \\ &= \mu \log_e\left(1+\frac{d}{n}\right) \\ &= \mu\left\{\frac{d}{n}-\frac{d^2}{2n^2}+\frac{d^3}{3n^3}- \ . \ . \ .\right\} \end{aligned}$$

But in the tables referred to, the logarithms of successive numbers of 5 digits are given, so that for the present purpose n may be considered to consist of 5 digits; and if d is small, viz. unity or a proper fraction, the terms of the right-hand side after the first will be exceedingly small, and may be neglected, so that $\log_{10}(n+d) - \log_{10} n = \frac{\mu}{n} d$.

Here μ is constant, and n is a given number,

$$\therefore \ \log_{10}(n+d)-\log_{10} n \propto d.$$

If $\quad d = 1, \log_{10}(n+1) - \log_{10} n = \frac{\mu}{n},$

i.e. D the tabular difference $= \frac{\mu}{n}$;

hence $\quad \log_{10}(n+d) - \log_{10} n = dD,$

$$\text{or } \log_{10}(n+d) = \log_{10} n + dD,$$

which is the formula to be practically applied.

Conversely, for finding a number corresponding to a given logarithm not exactly found in the tables; if δ be the excess of the given logarithm above $\log n$, and the number required be $n+x$, we have

$$\begin{aligned} \delta = \log_{10}(n+x) - \log_{10} n &= \frac{\mu}{n} x \\ &= Dx. \\ \therefore \ x &= \frac{\delta}{D}. \end{aligned}$$

Exercise 68.

(1) Find $\log_e 2$ and $\log_e 3$ to six decimal places.

(2) With the modulus given in (166) find $\log_{10} e$, and $\log_{10} e^2$.

(3) Having given $\log_{10} 150 = 2 \cdot 1760913$, find $\log_{10} 151$.

(4) Find $\log_{10} 101$ to seven decimal places.

(5) By making $m = x^2$, $n = x^2 - 1$, in the formula for $\log_e \frac{m}{n}$ in (165) show that

$$\log_e (x+1) = 2 \log_e x - \log_e (x-1) - 2\left\{\frac{1}{2x^2-1} + \frac{1}{3}\frac{1}{(2x^2-1)^3} + \dots\right\}$$

and thence, having given $\log_{10} 2 = \cdot 3010300$, $\log_{10} 3 = \cdot 4771213$, find $\log_{10} 82$ and $\log_{10} 79$.

CHAPTER XXXII.

INTEREST AND ANNUITIES.

(See *Arithmetic*, ch. xi.)

169.—The meaning of the terms *Principal*, *Interest*, &c. will be already understood from Arithmetic, the only alteration which will be made in their use here being that instead of a rate *per cent.* which is convenient practically in arithmetic, it is simpler in algebra to employ a symbol (r) to indicate the interest on *one* pound for a year.

The principal will be represented by P,
interest on £1 for a year . . . . r,
amount of £1 in a year . . . . . R,
number of years . . . . n,
amount . . . . M.

Then the following relations will be readily seen to subsist between the symbols:—

$$R = 1 + r \quad \ldots\ldots \quad \text{(i)}$$

$$\text{Simple interest on } P \text{ for a year} = Pr \quad \ldots\ldots \quad \text{(ii)}$$

$$\text{Amount} \ldots\ldots = PR \quad \ldots\ldots \quad \text{(iii)}$$

$$\text{Simple interest on } P \text{ for } n \text{ years} = Pnr \quad \ldots\ldots \quad \text{(iv)}$$

$$\text{Amount } (M) \ldots\ldots = P(1 + nr) \quad \ldots \quad \text{(v)}$$

From (v) any three of the quantities P, n, r, M, being given, the fourth may be found.

170.—Since P will in n years amount to M, P paid at the present time may be considered as *equivalent in value* to M due at the end of n years, so that we may either regard M as the amount of the given present sum P,

or P as the present value of a given future sum M.

171.—*If Compound* Interest is reckoned, and the interest *is payable yearly,*

the amount in one year will be $P(1+r) = PR$,
two years . . . . $PR(1+r) = PR^2$,
three . . . . $= PR^3$,

and generally, $M = PR^n$ . . . . (vi)

Hence also $P = \dfrac{M}{R^n}$ . . . . (vii)

If compound interest is reckoned, payable half-yearly,

$$\text{the amount for } \tfrac{1}{2} \text{ year} = P\left(1 + \frac{r}{2}\right),$$

$$1 \text{ year} = P\left(1 + \frac{r}{2}\right)^2,$$

$$1\tfrac{1}{2} \text{ year} = P\left(1 + \frac{r}{2}\right)^3,$$

&c.

$$n \text{ years} = P\left(1 + \frac{r}{2}\right)^{2n},$$

Similarly, if the interest is due q times a year,

$$M = P\left(1 + \frac{r}{q}\right)^{nq} \ldots\ldots\ldots \text{(viii)}$$

If the last formula be expanded,

$$M = P\left\{1 + nq\cdot\frac{r}{q} + \frac{nq(nq-1)}{1 \cdot 2}\cdot\left(\frac{r}{q}\right)^2 + \ \ldots\ \right\}$$

$$= P\left\{1 + nr + \frac{n\left(n - \frac{1}{q}\right)}{1 \cdot 2}r^2 + \frac{n\left(n - \frac{1}{q}\right)\left(n - \frac{2}{q}\right)}{1 \cdot 2 \cdot 3}r^3 + \ldots\right\}$$

Now suppose the interest to become due at indefinitely small intervals, *i.e.* every instant; then q becomes indefinitely large, and the fractions $\frac{1}{q}, \frac{2}{q}$, &c. may be neglected. In this case

$$M = P\left\{1 + nr + \frac{n^2r^2}{1 \cdot 2} + \frac{n^3r^3}{1 \cdot 2 \cdot 3} + \ \ldots\ \right\}$$

$$= Pe^{nr} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ix)}$$

Exercise 69.

(1) Obtain the values of r and R, for the rates 5, 4, 3, $5\frac{1}{4}$, *per cent.*; and the rates per cent. when $r = \cdot 02$, $\frac{3}{50}$, $\cdot 0\dot{5}$; and when $R = \frac{7}{6}$, $1 \cdot 045$, $1 \cdot 08\frac{1}{3}$.

(2) £A amount in n years to £B, find r and the rate per cent. (i) if simple interest is reckoned, (ii) if compound.

(3) In how many years will £A amount to £B (i) at simple interest, (ii) at compound interest due yearly, (iii) at compound interest due every instant; using r and R in their usual sense?

(4) Find the difference (to five places of decimals) between the amount of £1 in two years at 6 per cent. per annum compound interest, according as the interest is due yearly or monthly. (*Use logarithms.*)

(5) What is the true annual rate of interest in the formulæ (viii) and (ix) above?

172.—An *Annuity* is a sum which is payable yearly: it may be due either all at once each year, or in parts at fixed periods in the year. Unless specially expressed, the former is understood to be the case.

In computing annuities compound interest must be reckoned.

An annuity will be represented by A.

173.—An annuity which has been left unpaid for n years will at the end of that time amount to

$$\frac{R^n - 1}{R - 1} A.$$

For the sum (A) which was due at the end of the first year has been left unpaid for $(n-1)$ years, and by (vi) amounts in that time to AR^{n-1};

A due at the end of the 2nd year amounts to AR^{n-2};

&c.

Hence for the whole amount,

$$M = A(R^{n-1}+R^{n-2}+\ldots\ldots+R+1)$$

$$= \frac{R^n-1}{R-1}A \quad \ldots\ldots\ldots \text{ by (100).}$$

174.—The *Present Value* of an annuity to be paid for n years (commencing at the end of the first) is

$$\frac{R^n-1}{R-1}\cdot\frac{A}{R^n}.$$

For its value at the *end* of n years is $\frac{R^n-1}{R-1}A$,

the present value of which is by (vii)

$$\frac{R^n-1}{R-1}\cdot\frac{A}{R^n}.$$

Or the present value may be found in this way:

Present value of A due in 1 year $= \frac{A}{R}$,

A ... 2 years $= \frac{A}{R^2}$.

&c.

A ... n years $= \frac{A}{R^n}$.

$\therefore$ taking the sum of all,

$$P = \frac{A}{R}+\frac{A}{R^2}+\ \ldots\ldots + \frac{A}{R^n}$$

$$= \left(R^{n-1}+R^{n-2}+\ldots+1\right)\frac{A}{R^n}$$

$$= \frac{R^n-1}{R-1}\cdot\frac{A}{R^n}.$$

If the annuity is to be *perpetual*, the number of terms in the above series is unlimited; and then by (101)

$$P = \frac{\frac{A}{R}}{1-\frac{1}{R}} = \frac{A}{R-1} \text{ or } \frac{A}{r}.$$

175.—In either of the ways adopted above, it may be

proved that if an annuity is to commence at the end of p years and to continue q years,

$$P = \frac{R^q - 1}{R - 1} \cdot \frac{A}{R^{p+q}}.$$

If it is to commence at the end of p years and to be perpetual,

$$P = \frac{A}{(R-1)R^p}.$$

176.—If the present value of an annuity be mA, it is said to be worth *m years' purchase.*

An annuity which is to continue for a fixed number of years is said to be *certain*: a *life* annuity is one which is to continue during a person's life.

The calculation of life annuities is effected on the same general principles as above, but in order to determine the number of years for which the annuity may be expected to be payable, it is necessary to employ tables based upon observation, which show for any age the probable duration of life.

177.—The calculation of *life insurances* depends upon the same principles, but applied inversely. In order that a certain sum may be secured, to be payable at the death of a person, he pays yearly a fixed *premium,* and it becomes necessary to determine the amount of this premium. It may be payable either for a definite period, or for the whole of his life.

If A represent the premium to be paid for n years to insure an amount M to be paid immediately after the last premium, we have by (173)

$$M = \frac{R^n - 1}{R - 1} A.$$

$$\therefore \quad A = \frac{M(R-1)}{R^n - 1} \text{ or } \frac{Mr}{R^n - 1}.$$

If M is to be paid a year after the last premium, then

$$M = \frac{R^n - 1}{R - 1} AR,$$

$$\text{and} \quad A = \frac{Mr}{R(R^n - 1)}.$$

Exercise 70.

(*Use logarithms wherever advantageous.*)

(1) At 5 per cent. per annum find the amount of an annuity which has been left unpaid for 4 years.

(2) Find the present value of an annuity of £20 for 5 years, reckoning interest at 4 per cent.

(3) What annuity for 10 years may be bought for £1000, the annual rate of interest being $\frac{1}{80}$ of the principal?

(4) A perpetual annuity of £250 is to be bought, to commence at the end of 10 years: reckoning interest at $3\frac{1}{2}$ per cent., what should be paid for it?

(5) Find the present value of an annuity of £200 payable quarterly, to begin at the end of 2 years, and to continue 10 years, at 4 per cent. per annum interest.

(6) Find the perpetual annuity which, commencing at the end of one year, will be equivalent to a perpetual annuity (A) which is to commence at the end of n years.

Ex. To an annuity of £100 to commence after 8 years, interest at 5 per cent.

(7) A debt of £185 is discharged by two payments of £100 at the end of one and two years: what rate of interest is paid?

(8) What sum should be paid for 10 years in order to secure an annuity of £300 to be paid for the next 10 years, or for the next 20 years; interest at 4 per cent.?

(9) Reckoning interest at 4 per cent., what annual premium should be paid for 30 years in order to secure £2000 to be paid at the end of that time; the premium being due at the commencement of each year?

(10) £30 annual premium is paid to a life insurance society for insuring £1000; at 4 per cent. interest, for how many years must the premium be paid that the society may sustain no loss?

CHAPTER XXXIII.

CONTINUED FRACTIONS.

(*Arithmetic*, chap. xix.)

178.—The term *Continued Fraction*, though applicable to any fraction of the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+}}}\ \&c.$$

is commonly restricted to one whose numerators are each = 1, viz.

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+}}}\ \&c.$$

For the arithmetical application of continued fractions the following propositions are needed.

179.—Prop. *Any proper fraction in its lowest terms may be converted into a terminated continued fraction.*

Let $\frac{b}{a}$ be a fraction, in which $b<a$;

$$\text{then } \frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{1}{p+\frac{c}{b}}$$

(if p be the quotient, and c the remainder, of $a \div b$)

$$= \frac{1}{p+\frac{1}{\frac{b}{c}}} = \cfrac{1}{p+\cfrac{1}{q+\cfrac{d}{c}}} \quad (\text{if } b=qc+d)$$

$$= \cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+}}}\ \&c.$$

The successive steps of the division are the same as those for finding the G. C. M. of a and b; and since these are prime to one another, there will be at length a remainder 1, and the fraction will then terminate.

The continued fraction may be more conveniently written as $\frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{r+}$ &c.

180.—The fractions formed by taking one, two, three, &c. of the quotients $p, q, r, \ldots$ are called the *convergent* fractions, and will be proved to be alternately greater and less than the true value, and to approach successively more nearly to it.

Prop. *The successive convergents are alternately* $>$ *and* $<$ *the true value.*

Let A be the true value of $\frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{r+}$ &c.;
then $p, q, r, \ldots$ being all positive integers,

$$p < p + \frac{1}{q+} \text{ \&c.}$$

$$\therefore \frac{1}{p} > \frac{1}{p + \frac{1}{q+} \text{ \&c.}} \qquad \text{i.e.} > A.$$

Again,

$$q < q + \frac{1}{r+} \text{ \&c.}$$

$$\therefore \frac{1}{q} > \frac{1}{q + \frac{1}{r+} \text{ \&c.}}$$

$$\therefore \frac{1}{p+\frac{1}{q}} < \frac{1}{p + \frac{1}{q+} \text{ \&c.}} \qquad \text{i.e.} < A.$$

and so on.

181.—**Prop.** *If* $\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}$, *be any three consecutive convergents;* m_1, m_2, m_3, *the quotients which produced them; then will*

$$\frac{u_3}{v_3} = \frac{m_3 u_2 + u_1}{m_3 v_2 + v_1}.$$

For the first three quotients being p, q, r,

the first three convergents are $\frac{1}{p}, \quad \frac{1}{p+\frac{1}{q}}, \quad \frac{1}{p+\frac{1}{q+\frac{1}{r}}} \ldots$ (i)

or when simplified, $\frac{1}{p}, \quad \frac{q}{pq+1}, \quad \frac{qr+1}{(pq+1)r+p} \ldots$ (ii)

In (i) it will be noticed that the second convergent is formed from the first by writing in it $p+\frac{1}{q}$ for p; and the third from the second by writing $q+\frac{1}{r}$ for q; and this will be the way in which *any* convergent will be formed from the preceding one.

Hence, $\frac{u_3}{v_3}$ will be formed from $\frac{u_2}{v_2}$ by using $m_2+\frac{1}{m_3}$ instead of m.

In (ii) a further law of formation is observable; for the third convergent has its

numerator $= r$ (2nd numerator) $+$ (1st numerator),

and its

denominator $= r$ (2nd denominator) $+$ (1st denominator).

And if the fourth convergent be obtained, it will be found that its

numerator $= s$ (3rd numerator) $+$ (2nd numerator),

and its

denominator $= s$ (3rd denominator) $+$ (2nd denominator).

Now assume that this law holds up to the three consecutive convergents $\frac{u_0}{v_0}, \frac{u_1}{v_1}, \frac{u_2}{v_2}$: so that

$$u_2 = m_2u_1+u_0, \qquad v_2 = m_2v_1+v_0;$$

then since by (i) $\frac{u_3}{v_3}$ is formed from $\frac{u_2}{v_2}$ by having $m_2+\frac{1}{m_3}$ written instead of m_2,

$$\frac{u_3}{v_3} = \frac{\left(m_2+\frac{1}{m_3}\right)u_1+u_0}{\left(m_2+\frac{1}{m_3}\right)v_1+v_0} = \frac{m_3(m_2u_1+u_0)+u_1}{m_3(m_2v_1+v_0)+v_1}$$

$$= \frac{m_3u_2+u_1}{m_3v_2+v_1}.$$

Therefore the law will hold for $\frac{u_3}{v_3}$; and being shown above to be true for the third convergent, will be true for all.

182.—PROP. *The difference between two successive convergents* $\frac{u_1}{v_1}$ *and* $\frac{u_2}{v_2}$ *is* $\frac{1}{v_1v_2}$.

This proposition is true for the first two convergents, for

$$\frac{1}{p} - \frac{q}{pq+1} = \frac{pq+1-pq}{p(pq+1)} = \frac{1}{p(pq+1)}.$$

Assume it to be true for $\frac{u_0}{v_0}$ and $\frac{u_1}{v_1}$, so

$$\text{that } \frac{u_0}{v_0} \sim \frac{u_1}{v_1} = \frac{1}{v_0v_1};$$

$$\begin{aligned} \text{then } \frac{u_2}{v_2} \sim \frac{u_1}{v_1} &= \frac{u_2v_1 \sim u_1v_2}{v_1v_2} \\ &= \frac{(m_2u_1+u_0)v_1 \sim u_1(m_2v_1+v_0)}{v_1v_2} \\ &= \frac{u_0v_1 \sim u_1v_0}{v_1v_2} \\ &= \frac{1}{v_1v_2} \text{ by the assumption.} \end{aligned}$$

Hence being true for any one pair of consecutive convergents, the proposition will be true for the next pair: and having been shown to be true for the first pair, will be always true.

183.—Since the real value A lies between two successive convergents $\frac{u}{v}$ and $\frac{u_1}{v_1}$, $\frac{u}{v}$ will differ from A by a quantity less than $\frac{u}{v} \sim \frac{u_1}{v_1}$, *i.e.* $< \frac{1}{vv_1}$; so that the error made by taking $\frac{u}{v}$ for A is $< \frac{1}{vv_1}$, and *à fortiori* $< \frac{1}{v^2}$.

184.—PROP. *Any convergent* $\frac{u}{v}$ *will be in its lowest terms.*

For if u, v, had any common factor, it would also be a factor of $uv_1 \sim u_1v$, *i.e.* of 1.

185.—PROP. *The successive convergents approach more nearly to the true value* A.

For any convergent $\frac{u_2}{v_2}\left(=\frac{1}{p+}\,\frac{1}{q+}\cdot\;\cdot+\frac{1}{m_1+}\frac{1}{m_2}\right)$

differs from $\qquad A\left(=\frac{1}{p+}\,\frac{1}{q+}\cdot\;\cdot+\frac{1}{m_1+}\frac{1}{m_2+}\frac{1}{m_3}\cdots\right)$

by having m_2 used instead of $m_2+\frac{1}{m_3+}\;\ldots$

$$= m_2+\frac{1}{\mu} \text{ suppose,}$$

where $\mu > m_3 > 1$.

$$\therefore A = \frac{\left(m_2+\frac{1}{\mu}\right)u_1+u_0}{\left(m_2+\frac{1}{\mu}\right)v_1+v_0}$$

$$= \frac{\mu(m_2u_1+u_0)+u_1}{\mu(m_2v_1+v_0)+v_1}$$

$$= \frac{\mu u_2+u_1}{\mu v_2+v_1}.$$

$$\therefore A \sim \frac{u_2}{v_2} = \frac{\mu u_2+u_1}{\mu v_2+v_1} \sim \frac{u_2}{v_2}$$

$$= \frac{u_1v_2 \sim u_2v_1}{v_2(\mu v_2+v_1)}$$

$$= \frac{1}{v_2(\mu v_2+v_1)}.$$

$$\text{And } \frac{u_1}{v_1} \sim A = \frac{u_1}{v_1} \sim \frac{\mu u_2+u_1}{\mu v_2+v_1}$$

$$= \frac{\mu}{v_1(\mu v_2+v_1)}.$$

Now $1 < \mu$, and $v_2 > v_1$, for both which reasons

$$A \sim \frac{u_2}{v_2} < \frac{u_1}{v_1} \sim A,$$

i.e. a convergent $\frac{u_2}{v_2}$ is nearer to A than $\frac{u_1}{v_1}$ is.

186.—Prop. *Any convergent $\frac{\mathrm{u}}{\mathrm{v}}$ is nearer to the true value* A *than any other fraction with a smaller denominator.*

For let $\frac{x}{y}$ be a fraction in which $y < v$.

If $\frac{x}{y}$ be one of the convergents, it is shown in the last *proposition to be* farther from A than $\frac{u}{v}$ is.

If it be not one of the convergents, and be nearer to A than $\frac{u}{v}$ is, it must $\left(\text{since } A \text{ lies between } \frac{u}{v} \text{ and } \frac{u_1}{v_1}\right)$ be nearer to $\frac{u_1}{v_1}$ than $\frac{u}{v}$ is; *i.e.*

$$\frac{x}{y} \sim \frac{u_1}{v_1} < \frac{u}{v} \sim \frac{u_1}{x_1},$$

$$\text{or} \quad \frac{v_1 x \sim u_1 y}{v_1 y} < \frac{1}{v v_1},$$

and since $y < v$, this would require $v_1 x \sim u_1 y$ to be < 1; which is impossible, since x, y, u_1, v_1 are integers.

187.—The value of a continued fraction in which the quotients recur may be obtained by a quadratic equation.

Ex. To find the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{a} + \frac{1}{b} + \&c.$

Let x be the value, then

$$x = \cfrac{1}{a + \cfrac{1}{b+x}} = \frac{b+x}{ab+ax+1};$$

$$\therefore\ ax^2 + abx = b,$$

from which x may be found.

Note.—a and b being positive, (140. vi) shows that the roots will be real.

188.—A quadratic surd may be expressed in the form of a continued fraction, as in the following example.

Ex. To find $\sqrt{3}$ as a continued fraction.

The greatest integer in $\sqrt{3}$ is obviously 1.

Suppose $\qquad \sqrt{3} = 1 + \frac{1}{a},$

then $\qquad \frac{1}{a} = \sqrt{3} - 1;$

$$\therefore\ a = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2} \text{ by (131)}$$

$$= 1 + \frac{1}{b};$$

$$\therefore\ \frac{1}{b} = \frac{\sqrt{3}+1}{2} - 1 = \frac{\sqrt{3}-1}{2};$$

$$\therefore\quad b = \frac{2}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{1}$$

$$= 2+\frac{1}{c};$$

$$\therefore\quad \frac{1}{c} = \sqrt{3}+1-2 = \sqrt{3}-1,$$

$$c = \frac{1}{\sqrt{3}-1} = a \text{ above};$$

$$\therefore\quad \sqrt{3} = 1+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\ \&c.$$

or
$$1+\frac{\dot{1}}{1+}\,\frac{\dot{1}}{2};$$

the convergents of which will be found to be

$$1,\ 2,\ \frac{5}{3},\ \frac{7}{4},\ \frac{19}{11},\ \frac{26}{15},\ \frac{71}{41},\ \&c.$$

Exercise 71.

(1) Find the continued fractions equal to

$$\frac{31}{75},\ \frac{123}{157},\ \sqrt{5},\ \sqrt{11},\ 4\sqrt{6};$$

and the fifth convergent to each.

(2) Verify (182) by the convergents in the first and last of the above examples.

(3) Find the proper fraction which, when converted into a continued fraction, will have quotients 1, 7, 5, 2.

(4) Two convergents being $\frac{3}{14}$ and $\frac{19}{89}$, and the next quotient 5, find the next convergent.

(5) Find a fraction with denominator $<$ 100 which shall differ from $\sqrt{2}$ by less than $\frac{1}{10000}$.

(6) Find the surd values of

$$3+\frac{\dot{1}}{1+}\,\frac{\dot{1}}{6},\quad \frac{\dot{1}}{3+}\,\frac{1}{1+}\,\frac{\dot{1}}{6},\quad \text{and } 1+\frac{\dot{1}}{2+}\,\frac{1}{3+}\,\frac{\dot{1}}{4}.$$

MISCELLANEOUS EXAMPLES.

Exercise 72.

(1) If $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ show that $\frac{ma^2+nb^2}{ma^2-nb^2} = \frac{mc^2+nd^2}{mc^2-nd^2}$.

(2) If $s = \frac{a+b+c}{2}$ find the value of $\sqrt{s(s-a)(s-b)(s-c)}$ in terms of a, b, c, alone.

If a, b, c, are the sides of a right-angled triangle, show that $\sqrt{s(s-a)(s-b)(s-c)}$ is equal to the area of the triangle.

(3) Reduce to its simplest form the expression

$$\frac{(yz-ax)^2-(ca-y^2)(ab-z^2)}{(bc-x^2)(yz-ax)-(zx-by)(xy-cz)}.$$

(4) Extract the cube root of

$$x^9-3x^7-3x^6+3x^5+6x^4+2x^3-3x^2-3x-1.$$

(5) Solve the equations—

(i) $\frac{1}{x-a}+\frac{1}{x-b} = \frac{1}{x+a}+\frac{1}{x+b}$.

(ii) $\frac{x^4+a^2x^2+a^4}{x^2+ax+a^2}+\frac{x^4+a^2x^2+a^4}{x^2-ax+a^2} = 2b^2$.

(6) Solve—

$$\begin{cases} lx+my+nz = a, \\ l^2x+m^2y+n^2z = a^2, \\ l^3x+m^3y+n^3z = a^3. \end{cases}$$

(7) Two vessels, A and B, contain each a mixture of water and wine, A in the ratio of $2:5$; B in that of $3:11$. What quantity must be taken from each to form a mixture which shall consist of 5 gallons of water and 13 of wine?

(8) Solve the equations—

(i) $(x+9)^{\frac{1}{2}}+(x-7)^{\frac{1}{2}} = 8.$

(ii) $\dfrac{(x-a)^2}{x-b}-\dfrac{(x-b)^2}{x-a}=\dfrac{b^2}{a}-\dfrac{a^2}{b}.$

(9) If α, β, are the roots of the equation $x^2-px+q=0$, show that $\alpha+\beta=p$, $\alpha\beta=q$; and obtain in terms of α, β, the equation whose roots are p and q.

(10) Two trains starting from the termini A and B of a railway meet at a point $12\frac{1}{2}$ miles nearer A than B: had each travelled 5 miles an hour slower, they would have met half an hour later 15 miles nearer A than B. Find the length of the line.

(11) Which is the greater $\dfrac{6}{6+\sqrt{3}}$ or $\dfrac{2}{2\sqrt{3}-1}$, and by how much? Find the square root of $9-\frac{3}{2}\sqrt{35}$.

(12) If $a^x+a^{-x}=6$, find a^x-a^{-x}.

(13) If a, b, c, are proportionals, show that

$$(a-b)^2 : (b-c)^2 :: a : c,$$

and that $a+c > 2b$.

(14) Solve $4x^3+24x^2+47x+30=0$, its roots being in arithmetical progression.

(15) There are three numbers in geometrical progression; and if 1 be taken from the second and 5 from the third, or if the first be multiplied by 4, the second by 3, and the third by 2, they will be in arithmetical progression: find them.

(16) Expand $\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)^7$; and $(x^2-y^2)^{\frac{1}{7}}$ to four terms. Write down the $(r+1)$th term of the latter.

(17) Write down the quotients of $1-16x^4$ by $1+2x$, and $a+x$ by $\sqrt[3]{a}+\sqrt[3]{x}$.

(18) Resolve

$a^{-3}-8$, $2x^{-2}-16x^{-1}+24$, and $3(m^2-n^2)^{-1}+5(m+n)^{-2}$

into *their simplest factors* with positive indices.

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(19) Find the square root of

$$1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2};$$

and of $17+12\sqrt{2}$.

(20) Reduce to its simplest terms

$$\frac{a^{-1}+b^{-1}+c^{-1}}{1-a^{-1}b^{-1}-a^{-1}c^{-1}-b^{-1}c^{-1}}, \text{ when } a^{-1} = -\frac{b^{-1}+c^{-1}}{1-b^{-1}c^{-1}}.$$

(21) Solve

$$\sqrt{\frac{3x}{x+y}}+\sqrt{\frac{x+y}{3x}} = 2, \quad xy-(x+y) = 54.$$

(22) The combinations of n things taken r, $r+1$, $r+2$, together are proportional to 3, 8, 14; find n and r.

(23) Solve $7x+11y = 145$ in positive integers; and find the least integer which, when divided by 5, 7, 11 respectively, will have remainders 3, 4, 7.

(24) Expand $(a^2-ax)^{-\frac{1}{2}}$ to five terms; and show that the $(r+1)$th term

$$= \frac{\underline{|2r}}{(\underline{|r}\,.\,2^r)^2}\, a^{-1-r}x^r.$$

(25) An officer can form his men into a hollow square 4 deep, or into a hollow square 8 deep, and the front in the latter formation contains 16 men fewer than in the former. Find the number of men.

(26) Find the price of eggs per score when 10 more in half-a-crown's worth lowers the price 1*s.* 3*d.* per hundred.

(27) If

$$f(m) = 1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+ \ldots$$

show by multiplication to four terms that

$$f(m)\times f(n) = f(m+n).$$

(28) Write down the 5th, 11th, and middle terms of $(a+b)^{14}$; and the term involving x^7 in the expansion of $\left(x^2-\frac{a^3}{x}\right)^8$.

(29) Solve $a^{2x}-8a^x+7=0$; and $2^{2x}=2^{2+x}+32$.

(30) Find the present value of an annuity of £100 for 12 years; interest 4 per cent. per annum.

(31) Simplify $\{x^{p-q}-x^{q-r}\}\cdot\{x^{p+r}+x^{2q}\}\div\{x^{2p-q}-x^{3q-2r}\}$.

(32) Find the relation between a, b, c when x^4+ax+b is exactly divisible by $x+c$.

(33) If the geometrical mean between a and b : harmonical mean : : $m : n$, find the ratio of $a : b$.

(34) Simplify—

(i) $$\left\{\frac{x+b}{x^2+2ax-bx-2ab}-\frac{x+2a}{x^2+bx-2ax-2ab}\right\}\div\frac{3x^2+bx+2ax-2ab}{x^2+2ax+bx+2ab}.$$

(ii) $$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2+\sqrt{7+3\sqrt{5}}}}-\frac{\sqrt{5}-\sqrt{2}}{\sqrt{2+\sqrt{7-3\sqrt{5}}}}.$$

(35) Show that the roots of the equation
$$(x-m)(x-n)=mnx^2$$
will be real if m and n are real.

(36) If a, b, c are proportionals, $(a^2-b^2+c^2)>(a-b+c)^2$.

(37) Solve—

(i) $$\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}=\frac{x^2+4x+6}{x+2}+\frac{x^2+6x+12}{x+3}.$$

(ii)
$$\sqrt{a(a+bx)}+\sqrt{b(b-ax)}+\sqrt{ab(a+bx)(b-ax)}=\sqrt{a^2+b^2}.$$

(iii) $7^{\frac{x}{2}+\frac{y}{3}}=2401$; $6^{\frac{x}{4}+\frac{y}{2}}=1296$.

(38) The diameter of a cube is a foot longer than its side; find the side, and the area of the section through two opposite edges.

(39) A contractor undertook to complete some work in 210 days, and engaged 15 men to do it. But after 100 days he found it necessary to engage 10 men more, and then he accomplished the work 5 days too soon. How many days behindhand would he have been if he had not engaged the 10 additional men?

(40) If $\frac{1}{x-b}-\frac{1}{x-a}=\frac{4}{a-b}$, x will be the arithmetical mean between a and b.

(41) If a, b, c, are the pth, qth, and rth terms of a geometrical series, $a^{q-r}\,.\,b^{r-p}\,.\,c^{p-q}=1$.

(42) A and B, who are 91 miles apart, set out to meet one another. A goes 15 miles the first day, 13 the second, 11 the third, and so on. B goes 1 mile the first day, 2 the second, 3 the third, and so on. In how many days will they meet? Explain the double answer.

(43) The sum of three numbers in harmonical progression is 13, and the difference between the greatest and least is three. What are the numbers?

(44) Show by an inductive proof that

$$1^3+2^3+3^3+\ .\ .\ .\ +n^3=(1+2+3+\ .\ .\ .\ +n)^2.$$

(45) One root of the equation $6x^{\frac{3}{2}}-5x-17x^{\frac{1}{2}}+6=0$ is 4; find the other roots.

(46) Solve the equation

$$(x-1)(x-2)+(x-2)(x-3)+(x-3)(x-4)=2;$$

and thence solve

$$(x^2-x-1)(x^2-x-2)+(x^2-x-2)(x^2-x-3)$$
$$+(x^2-x-3)(x^2-x-4)=2.$$

(47) Half of the contents of a vessel full of wine are drawn off, and the vessel is then filled up with water. Find the quantity of wine remaining in the vessel after this process has been repeated n times, and show that the quantity of wine drawn off will be to the quantity of water drawn off, as

$$2^n-1\ :\ (n-2)\,2^{n-1}+1.$$

(48) Find the sum of the square roots of the roots of the equation $x^2-x+16=0$.

(49) Sum the following series:—

(i) $\frac{1}{2+\sqrt{2}}+2+\frac{5-2\sqrt{2}}{2-\sqrt{2}}+$ &c. to eight terms.

(ii) $1+2r+3r^2+4r^3+\ .\ .\ .$ to n terms.

(50) Expand $(a^2-x^2)^{-\frac{3}{2}}$ to 5 terms, and show that the middle term in the expansion of $\left(x+\frac{1}{2x}\right)^{2n}$ is equal to

$$\frac{\lfloor 2n}{2^n.\left(\lfloor n\right)^2} \text{ or } \frac{1.3.5\ldots\ldots(2n-1)}{1.2.3\ldots\ldots n}.$$

(51) Solve the equations—

(i) $\frac{x^2}{a}-3ax=\sqrt{4x^3+9ax^2}+\frac{27a^2}{4}$.

(ii) $x^4-3x^3-3x^2+7x+6=0$.

(iii) $\left.\begin{aligned} xy+yz&=8\\ xy+xz&=5\\ z^2-xy&=7\end{aligned}\right\}$

(52) In how many different orders may a cricket eleven have their innings, the first two being considered to go in simultaneously.

(53) Resolve $\frac{12x-1}{x(x^2-1)}$ into its partial fractions.

(54) Which is the greater, $2\sqrt{2}-\sqrt{5}$ or $3\sqrt{2}-\sqrt{7}$?

(55) Simplify

$$\left\{\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)^{-\frac{1}{2}}-\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^{-\frac{1}{2}}\right\}\times\left\{\left(b^{-\frac{1}{2}}-a^{-\frac{1}{2}}\right)^{\frac{1}{2}}+\left(b^{-\frac{1}{2}}+a^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right\}.$$

(56) Extract the square root of $15-4\sqrt{14}$; and express as a single surd $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[4]{4}\times\sqrt{5}$.

(57) Form an equation whose roots shall be the squares of those of $2x^2+3x+4=0$.

(58) 17 books are contained in one shelf, of which 8 volumes form one set and 3 another, the rest being single. In how many ways may they be arranged, the volumes of each set being arranged from left to right?

(59) Expand $(3+4d)^{\frac{3}{4}}$ to 4 terms.

Supply the sign and index of $(1 \quad x)$ that the expansion may have its rth term always rx^{r-1}.

(60) What perpetual annuity, to begin at the end of 6 years, should be purchasable for £1000 if interest is *reckoned at 5 per cent.* per annum?

(61) Having given A the area of a right-angled triangle, find the sides containing the right angle when their sum is the least possible.

(62) If $y = x + 4x^2 + 7x^3 + 10x^4 \ldots$ find x in terms of y.

(63) If the number of permutations of 24 things taken r together be double of that of 23 things taken r together, find r.

(64) How many different numbers of four digits can be formed with three fours, two fives, and one six, and what would be the sum of all?

(65) Expand $(2x-3y)^{\frac{2}{5}}$ to four terms, and find the value of the sixth term when $x = 2\frac{2}{3}$, $y = \frac{4}{9}$.

(66) If $a, b, c \ldots$ are the digits of a number N, beginning from the unit's place, show that N is divisible by 6 if $a + 2^2b + 2^4c + 2^6d +$ &c. is so divisible.

(67) Resolve into their partial fractions

$$\frac{5}{(2x-1)(x^2-1)} \text{ and } \frac{9x^2y+5xy^2-y^3}{x^4+x^2y^2+y^4}.$$

(68) Find the fractions convergent to $\frac{341}{469}$. Express $\sqrt{13}$ in the form of a continued fraction, and write down its sixth convergent.

(69) Eliminate x, y, z, from the equations

$$x^2(y+z) = a^3, \quad y^2(z+x) = b^3, \quad z^2(x+y) = c^3, \quad xyz = abc.$$

(70) If $x = z - \frac{z^3}{3} + \frac{z^5}{5} -$ &c., and $y = z\sqrt{1-y^2}$, find y in a series of ascending powers of x.

(71) Solve the equations—

(i) $3^{x+1} - 3\sqrt{3} = 3^{2x-1} - 1$.

(ii) $(a+b)^2x + \left(\frac{ab}{a-b}\right)^2\left(4 - \frac{3}{x}\right) = 2ab$.

(iii) $x + y + \sqrt{xy} = 28$.
$x^2 + y^2 + xy = 336$.

(72) There is a number of two digits which, when divided by the sum of its digits, gives a quotient greater by 2

than the first digit; and if the digits be inverted, and divided by a number greater by 1 than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Find it.

(73) A person starts to walk at a uniform speed without stopping from C to D and back, at the same time that another person starts to walk at a uniform speed without stopping from D to C and back. They meet a mile and a half from D, and again, an hour after, a mile from C. Find the distance from C to D, and their rates of walking.

(74) If $S_1, S_2, S_3 \ldots S_n$ are the sums of n arithmetical progressions, each to n terms, whose first terms are $1, 2, 3, \ldots n$, and common differences $1, 3, 5, \ldots \overline{2n-1}$ respectively; show that

$$S_1+S_2+S_3+\ \ldots\ +S_n=\frac{n^2(n^2+1)}{2}.$$

(75) The first term of a geometrical progression is b^2-c^2, and the third is $(b+c)^2-6(b+c)c+12c^2-\dfrac{8c^3}{b+c}$: find the sum of the series to infinity.

(76) If P, Q be the mth and nth terms of a progression, show that the $\overline{m+n}$th term is equal to

$$\frac{mP-nQ}{m-n},\quad \left(\frac{P^m}{Q^n}\right)^{\frac{1}{m-n}},\ \text{or}\ \frac{PQ(m-n)}{mQ-nP},$$

according as the series is arithmetic, geometric, or harmonic.

(77) Sum to n terms the series

$$1\cdot\tfrac{1}{2}-2\cdot\frac{1}{2^2}+3\cdot\frac{1}{2^3}-\ldots\ldots$$

(78) What space of ground will be required for a square pyramid containing 385 10-inch shells?

(79) Prove that

$$(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4$$
$$=12abc(a+b+c).$$

(80) Reduce to its simplest form the expression

$$\frac{\sqrt{18}}{\sqrt{3}+\sqrt{2}}+\frac{\sqrt{12}}{\sqrt{3}-\sqrt{2}};$$

and extract the square roots of $14+4\sqrt{6}$, and $x+2\sqrt{x-1}$.

(81) In a race between two boats a spectator walking at the rate of 5 miles an hour is $\frac{1}{8}$ of a mile ahead of the first boat at starting, and when it passes him, the interval between the two boats, which was 30 yards at starting, is reduced to 10. At the distance of $1\frac{1}{4}$ mile from the place where it started, the first boat is overtaken by the second. Find the duration of the race.

(82) Solve (i)

$$\begin{aligned} x^2+y^2+z^2 &= 11, \\ xy+xz+yz &= -1, \\ x+y-z &= -3; \end{aligned}$$

and (ii)

$$\begin{aligned} \sqrt{x}+\sqrt{y}-\sqrt{z} &= 6, \\ \sqrt{xy}+\sqrt{xz}+\sqrt{yz} &= 11, \\ x+y-z &= 6. \end{aligned}$$

(83) A person rides from A to B and back at a uniform rate. But the clock at B being $3'$ slow, he appears, if he reckons his time at B by that, to have gone at the rate of $8\frac{1}{8}$ miles, and to have returned at the rate of $6\frac{9}{11}$ miles per hour. Find his actual rate and the distance from A to B.

(84) Show that a^5+b^5 is divisible by $a^2+abx+b^2$ if $(2x+1)^2=5$.

(85) Find the relation between a, b, c, d, in order that $ax^2+bxy=c$, $axy+by^2=d$, and $a^2(x+y)^2+b^2(x-y)^2=ac+bd$, may be satisfied by the same values of x and y. And show that if c and d have the same sign, a cannot be greater than b.

(86) If $x+c$ be the G.C.M. of x^2+ax+b and $x^2+a'x+b'$, their L.C.M. will be

$$x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c.$$

(87) Solve the equations

$$\sqrt{ax(ax+by)}+\sqrt{bx(bx-ay)}+\sqrt{(ax+by)(bx-ay)}$$
$$=x\sqrt{a^2+b^2} \text{ and } \sqrt{ax}+\sqrt{by}=\sqrt{xy}+\sqrt{ab}.$$

(88) Simplify $\sqrt[3]{2} \cdot \sqrt[4]{6} \cdot \sqrt[6]{32} \cdot \sqrt[9]{27} \cdot \sqrt[12]{96^{-1}}$, and $\sqrt{-8+2\sqrt{7}}$; and find as a continued fraction the square of $2 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+} \dots$

(89) If α and β are the roots of the equation $x^2+px+q=0$, show that the equation whose roots are $\frac{\alpha^3}{\beta^2}$ and $\frac{\beta^3}{\alpha^2}$ is

$$q^2x^2+(5pq^2-5p^3q+p^5)x+q^3=0.$$

(90) If the present worth of a sum of money due in 8 years is $\frac{4}{9}$ of its amount in the same time at the same rate, find the rate (i) at simple, (ii) at compound interest.

(91) If $a, b, c \dots$ are the coefficients of $(1+x)^n$ where n is a positive integer; find, by comparing $(1+x)^n.(1+x)^n$ with $(1+x)^{2n}$, $ab+bc+cd+ \dots$ in terms of n.

(92) If the value of diamonds varies as the square of their weight, and the square of the value of rubies varies as the cube of their weight; and a diamond of a carats is worth m times a ruby of b carats, both together being £c; find the value of a diamond and of a ruby, each weighing n carats.

(93) If $a, b, c \dots$ are in geometrical progression, find to n terms

$$(a^2+b^2)^{-1}+(b^2+c^2)^{-1}+(c^2+d^2)^{-1}+ \dots .$$

and $$(a^4+b^4)^{\frac{1}{2}}+(b^4+c^4)^{\frac{1}{2}}+(c^4+d^4)^{\frac{1}{2}}+ \dots .$$

(94) If $y = e^{\frac{1}{1-\log x}}$ and $z = e^{\frac{1}{1-\log y}}$; then $x = e^{\frac{1}{1-\log z}}$.

(95) Divide $\frac{208}{105}$ into three fractions with prime denominators; and $\frac{x}{x^4+3x^2+2}$ into its partial fractions.

(96) Prove that $(a+b-c)^3+(a+c-b)^3+(b+c-a)^3 > 3abc$.

(97) Solve $x^3-3x = a^3+\frac{1}{a^3}$; and find the value of $\frac{1}{5+}\frac{\dot{1}}{3+}\frac{1}{1+}\frac{\dot{1}}{2}$.

(98) Resolve into factors $2x^2-42xy-44y^2+68y-x-3$.

(99) Two lines a and b are divided in the same ratio: determine this ratio in order that the squares on one corresponding part of each with twice the rectangle contained by the other parts may be the least possible.

(100) From the formula $(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$ show that if C_r denote the number of combinations of m things r together, and c_r the combinations of n things r together, the combinations of $m+n$ things r together

$$= C_r + C_{r-1}c_1 + C_{r-2}c_2 + \ldots\ldots$$

Show that the last example in Ex. 49 is a particular case of this.

ANSWERS TO EXAMPLES.

Ex. 1.

(1) $2, 3, 1, 4, 1;\ 2, 1, 3, 1;\ 1, 3, 5.$
(2) $2x+3y;\ 2x-3y.$
(3) $5, 18, 0, 24, 21, 12, -36, 0, -54.$
(4) $9, 24, 7, 0.$
(5) $13;\ 3.$
(6) $8, 120, 1728, 24, 18, 12.$
(7) $18, 24.$
(8) $-4a^2bx,\ a^2bx.$
(9) $11, 7, 4, 3, 6.$
(10) $5a^3xy,\ 2bx^2y^2,\ bx^4.$
(11) $2, 1, 5\frac{5}{7}, 0.$
(12) $81, 162, 160, 2, 8, \frac{5}{21}.$
(13) $a^2b^3,\ 2ax^3y^2.$
(14) $\frac{2a}{3b}, \frac{3x}{7y}, \frac{2a+5x}{3b-2y}.$
(15) x^3, x^2y, xy^2, y^3
(16) $25.$ (17) $784.$ (18) $3\frac{1}{3}.$ (19) $\frac{1}{72}.$ (20) $17{\cdot}29$

Ex. 2.

(1) $19a;\ 8x;\ 11y.$
(2) $a^2b,\ -5xy^2;\ -9a^3bc.$
(3) $8a+2b+5c.$
(4) $x-2y.$
(5) $10a^2b-7ab^2+b^3.$
(6) $-5a^2x^2+6ax^3.$
(7) $-c^4+c+8.$
(8) $x^3-y^3.$
(9) $-2xy+4xz-8y^2-yz-z^2.$
(10) $12a^3x-9ax^3.$
(11) $6m^5-m^4n+2m^3n^2+3m^2n^3-4mn^4+5n^5.$
(12) $-9abxy-5b^2y^2.$

Ex. 3.

(1) $2a;\ -3a;\ 6a;\ 9a;\ -6a;\ -2a.$
(2) $a+b+2c,\ -ab-5bc+4ac.$
(3) $-4xy+2xz-y^2+5yz-z^2.$
(4) $ax^2+7abx-bx^2+b^2x+12b^3+x^3.$
(5) $-2x^3+3xy^2-y^3-14x^2+2xy-10y^2+6.$
(6) $-a^4-8ab^3+b^4.$
(7) $-2x^5+4x^4y-8xy^4-8y^5.$
(8) $a^2b^2-3a^2bc-3ab^2c-a^2c^2-abc^2-b^2c^2.$
(9) $2a+4b-9c+d.$
(10) $b-a;\ a^3+a^2b+6ab^2+b^3.$

Ex. 4.

(1) $6a^3b^3$; $20a^3x^3y$; $7a^3b^3x^2y^4$; $10x^5y^3z^5$.
(2) $-12ab^2c$; $-14a^3b^3c^3$; $ab^2c^3d^3$; $18a^4bc$.
(3) $-84x^3y$; $4a^4b^2x^2y^4$; $-12x^3y^4z$; $20a^4b^4c^5d^5$.
(4) $8a^3b-12a^2b^2+20ab^3$; $-15x^5y+10x^4y^2+35x^3y^3$
$-5x^2y^4$.
(5) $a^4x^3y-5a^3x^4y+a^2x^5y+2ax^6y$; $27a^6b^4-9a^4b^6+12a^3b^7$
$+3ab^9$.
(6) $2a^2+5ab+2b^2$; $6a^3b-23a^2b^2+20ab^3$; $-4a^5+16a^3b^2$
$-4a^2b^3-8ab^4$.
(7) $a^4+a^2b^2+b^4$; $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.
(8) x^4-16y^4; $81x^6y-27x^4y^3-9x^2y^5+3y^7$.
(9) $24x^4y^2z^3+12x^3y^3z^2-6x^2y^4z-3xy^5$; $15x^5+3x^3y^2$
$-25x^2y^3-5y^5$.
(10) $25a^7b-9a^5b^3+22a^4b^4-4a^3b^5+ab^7$.
(11) $4a^{13}y^3+8a^{11}y^4-64a^5y^7-128a^3y^8$.
(12) $6m^7n-18m^5n^3+18m^3n^5-6mn^7$.
(13) $3a^2b-2a^2c+6ab^2-4abc+3b^3-2b^2c$;
$20a^7b-10a^5b^3-29a^4b^4+4a^3b^5+12a^2b^6+4ab^7-b^8$.
(14) $24a^9b-26a^5b^5+4a^4b^6-6a^3b^7-5a^2b^8+4ab^9-3b^{10}$.
(15) $x^8+x^4y^4+y^8$.
(16) $32a^7b-8a^6b^2+8a^5b^3-6a^4b^4-2a^3b^5-a^2b^6+ab^7$.
(17) $x^5+ax^4-27a^2x^3-13a^3x^2+134a^4x+120a^5$.
(18) $531441a^{12}-b^{12}$.
(19) $-y^2z^2-3z^4$. (20) $3a^4b-a^3b^2-9a^2b^3$.

Ex. 5.

(1) $3a^2b^2$; $5xy^2$; $4a^4$; $6a^4x$.
(2) $-5x^3y$; $-4a^2c$; $2axy^4$.
(3) $2a^2-3ab+6b^2$; $-4xy^2+5x^2y+8x^3$.
(4) $\dfrac{x^3}{y}-2x^2+3xy-1$; $-3+2ab+a^2b^2$.
(5) $-z-2xz^2+5x^2yz^2-6x^3y^2+3x^3y^2z$.

(6) $a+b$; $a-b$. (7) $a+3b$; x^2-3xy.
(8) $a^2b+2ab^2+2b^3$; a^2-ab+b^2.
(9) $x^2-3xy-y^2$. (10) $4x^2+2xy+y^2$.
(11) $-8a^3+2a^2b-ab^2$.
(12) $27x^3+9x^2y+3xy^2+y^3$; $a^3-2a^2b+4ab^2-8b^3$.

(13) $1+x-2x^2$; $-1-3ab-13a^2b^2$.
(14) $x^4+2x^3y+2x^2y^2+xy^3$; x^2-4x+8.
(15) $x^3-3x^2y+3xy^2-y^3$. (16) $a^4+a^2b^2+3b^4$.
(17) $27x^2y-18xy^2-9y^3$. (18) $a^2+2ab+4b^2$; $8ab^3$ rem.
(19) $x^4+3x^3y+8x^2y^2-8y^4$.
(20) $4a^2+4ab+3b^2$. (21) x^9.
(22) a^3+2a^2+3a+4; $-5a+3$.
(23) $x^2+2xy+y^2$. (24) $2-6x+4x^2-2x^3$.

Ex. 6.

(1) $x, x^2, y, y^2, xy, xy^2, x^2y$;
$2, 5, 10, x, 2x, 5x, 10x, y, 2y, 5y, 10y, xy, 2xy, 5xy$.
(2) $a^3, a^2b, a^2c, ab^2, abc, b^2c$.
(3) x^2y; a^2bc^2; $4xy$. (4) bc; $4y^2$.
(5) a^3b^4; $12x^2y^3$; $30abx^3yz^2$. (6) $60lm^4n^3$; $120a^3b^3cd^4e^2$.
(7) $a^2x, a^3bx^2y^3$; $6x^2y^2, 120x^3y^2z$.
(8) $2, a, a^2, c, 2a, 2a^2, 2c, ac, a^2c, 2ac, 2a^2c$.
(9) xy^4z. (10) a^2, ab, ac, bc.
(11) $2xz^2, x^2z^2$. (12) $a^2b^3c^2, ab^4c^2, ab^3c^3$.
(13) $\frac{4a}{5c}$. (14) $\frac{3z^3}{4x^3y}$. (15) $\frac{2ab}{3yz}$.
(16) $\frac{a}{e}$. (17) 1. (18) $\frac{25a^2x^2}{2c^2z^2}$.
(19) $\frac{13a}{12}$. (20) $\frac{a^2c+ab^2+bc^2}{abc}$. (21) $\frac{a^3+b^3+c^3}{abc}$.
(22) $\frac{3x^2z+5y^3+2xy^2}{2xy^2z}$. (23) $\frac{8xy^3-xz^3}{2y^2z^2}$.
(24) $\frac{5acx}{3by^2}$. (25) $\frac{8ad}{3bc}$. (26) $a-2c+\frac{c^2}{6a}$.
(27) $x+z-\frac{2z^4}{a^2y}$. (28) $2c-4b-\frac{3bc}{a}$.
(29) $-3x^2+\frac{4a^2xz^2}{3by^2}-\frac{2xyz}{b}$. (30) $\frac{a^2c+a^2b+bc^2}{ac}$.
(31) $\frac{yz^3+x^2y^2-x^3z}{yz^3}$. (32) $\frac{a^2b+b^2c+ac^2}{abc}$.
(33) $\frac{24a^2bc}{5xyz}, \frac{12abc}{xy}, \frac{36a^2b^2c}{z}$.
(34) $\frac{2xy^3}{9ab^2c^3}, \frac{10x^2y^3z}{27a^2b^3c^4}, \frac{2xz}{27a^3b^2c^3y}$.

(35) $\dfrac{10bx^2-5ay^2-2byz}{12xyz}$. (36) $\dfrac{2a^2b-2ab^2-a^2c+b^2c}{a^2b^2c}$.

(37) $\dfrac{a^2+2ax}{x^2}$. (38) $-\dfrac{5c^2}{18ab}$.

(39) $-\dfrac{1}{6y^2z}$. (40) $\dfrac{5a^3-2}{4a^2}$.

Ex. 7.

(1) $a-b$. (2) $4a$, ac. (3) $60m$, $\dfrac{n}{60}$.

(4) $\dfrac{x}{2}$, $\dfrac{x}{2}+y$. (5) $am-bm$ yards.

(6) $\dfrac{a}{10}$, $\dfrac{a}{b}$ shillings. (7) $\dfrac{88a}{3}$.

(8) $20y-20x$; $\dfrac{n}{x}-\dfrac{n}{y}$ shillings.

(9) $\dfrac{12a}{n}$, $\dfrac{9a}{20n}$. (10) $2a+2b$ ft., ab sq. ft.

(11) $2ac+2bc$ sq. ft. (12) $\dfrac{ab}{3x}$; $\dfrac{aby}{3x}$ pence.

(13) $20a-bc+d-\dfrac{e}{12}$. (14) $2x+5$ pounds.

(15) $2x+4$, $4x-3$. (16) $50-2x$.

(17) a^2+5; b^2-a^2-5. (18) $\dfrac{n}{400}$.

(19) $ab+c$. (20) $16m-64$.

(21) $\dfrac{a}{12}$. (22) $b+c-a$.

(23) $\dfrac{x}{a}$. (24) $a+\dfrac{40}{n}$ shillings.

(25) $\dfrac{abc}{108}$, $\dfrac{abc-108}{bc}$ yards. (26) $\dfrac{ab}{c}$.

(27) $4a+6$; $5a$. (28) $\dfrac{ac-ab+d}{c}$.

(29) $10a+b$. (30) $x-\dfrac{13b}{5}+\dfrac{13c}{5}$ pounds.

Ex. 8. $x=$

(1) 4. (2) 2. (3) 5. (4) -2.

(5) -1. (6) 1. (7) -1. (8) -18.
(9) $5\frac{1}{2}$. (10) $\frac{4}{13}$. (11) 6. (12) 20.
(13) 5. (14) 12. (15) 3. (16) -5.
(17) $1\frac{1}{8}$. (18) -2. (19) -11. (20) $\cdot 9$.
(21) $\cdot 1$. (22) -100. (23) 10. (24) 3.
(25) 12. (26) 11. (27) $-\frac{1}{4}$. (28) $4\frac{1}{2}$.
(29) 15. (30) $\cdot 7$.

Ex. 9.

(1) 8. (2) 24. (3) 168. (4) 80.
(5) 8. (6) 9, 6. (7) 30*s*. (8) 30*s*.
(9) 16. (10) 5 fl. 6 cr. (11) 119.
(12) In $2\frac{1}{2}$ hrs.; 16 m. from 1st place.
(13) 10, 15, 35 yards. (14) 24*s*., 13*s*.
(15) 10*d*., 12*d*., 4*d*., 22*d*. (16) 9*s*., 18*s*., 27*s*., 36*s*.
(17) 18. (18) 812. (19) 8, 4.
(20) A, 12*s*. 6*d*.; B, 7*s*. 6*d*. (21) 39*s*.
(22) 19, 20, 21, 22. (23) 9, 19, 14.
(24) At 50 miles. (25) $16\frac{4}{11}$ min. past 3. (26) $26\frac{2}{3}$ days.
(27) 97. (28) 180. (29) 10. (30) In 20 sec.

Ex. 10.

(1) $2b$. (2) $a+b-2c$. (3) $3x-3y-z$.
(4) $a-b+c$. (5) $-2x$. (6) $-9b+14c$.
(7) $3a-b-c$. (8) $4x-y-z$. (9) $-b+10c$.
(10) $-5a$. (11) $5a+6b-7c$. (12) $6a-11b-c$.
(13) $-2x$. (14) $6bx+4by$. (15) $-c$.
(16) $-25x$. (17) $5a$. (18) a.
(19) $2bx+2ay$. (20) $2ax-3bx-ay$.
(21) $-21nx-14my$. (22) 0.
(23) $-3bx$. (24) $-6x$.

Ex. 11.

(1) $(2a-3b)-(4c-d)+(3e-2f)$;
$(2a-3b-4c)+(d+3e-2f)$.
(2) $(a-2x)+(4y-3z)-(2b-c)$;
$(a-2x+4y)-(3z+2b-c)$.
(3) $(a^5+3a^4)-(2a^3+4a^2)+(a-1)$;
$(a^5+3a^4-2a^3)-(4a^2-a+1)$.

(4) $-(3a+2b)+(2c-5d)-(e+2f)$;
$-(3a+2b-2c)-(5d+e+2f)$.
(5) $(ax-by)-(cz+bx)+(cy+az)$;
$(ax-by-cz)-(bx-cy-az)$.
(6) $(2x^5-3x^4y)+(4x^3y^2-5x^2y^3)+(xy^4-2y^5)$;
$(2x^5-3x^4y+4x^3y^2)-(5x^2y^3-xy^4+2y^5)$.
(7) $\{2a-(3b+4c)\}+\{d+(3e-2f)\}$.
$\{a-(2x-4y)\}-\{3z+(2b-c)\}$.
$\{a^5+(3a^4-2a^3)\}-\{4a^2-(a-1)\}$.
$-\{3a+(2b-2c)\}-\{5d+(e+2f)\}$.
$\{ax-(by+cz)\}-\{bx-(cy+az)\}$.
$\{2x^5-(3x^4y-4x^3y^2)\}-\{5x^2y^3-(xy^4-2y^5)\}$.
(8) $3(a-2b)-4(c-2d)$. (9) $a(x+y)-b(x-y)$.
(10) $2a(x-3y)+4b(z-x)-c(2x+3y)$.
(11) $5x(x-2y)-3y(y-3z)+z(x+2z)$.
(12) $(a-b)x+(2a+3)y+(4a-3b-2)z$.
(13) $(a-4b-2c)x-(a+2b+3c)y+(a+4b+5c)z$.
(14) $(a+c)x-(a-b)y-(b+c)z$.
(15) $3(4a-5c)x+2(6a+2b+3c)y-6(2b+c)z$.
(16) $(a+2b+3c)x-ay-(a+2c)z$.
(17) $3ax-2cy+(2b+3c)z$. (18) $2(a-b)x-(3b+c)y$.
(19) $x^3-(a-b+c)x^2-(ab-ac+bc)x+abc$.
(20) $(a+2b+3c)x+(a-3b-6c)y+(2a-b-3c)z$.

Ex. 12.

(1) $x^2+2xy+y^2$; $y^2-2yz+z^2$; $4x^2+4x+1$
$4a^2+20ab+25b^2$; $1-2x^2+x^4$.
(2) $9a^2x^2-24ax^3+16x^4$; $1-14a+49a^2$; $25x^2y^2+20xy+4$;
$a^2b^2+2abcd+c^2d^2$; $9m^2n^2-24mn+16$.
(3) $144+120x+25x^2$; $16x^2y^4-8xy^3z^2+y^2z^4$;
$9a^2b^2c^2-6ab^2c^2d+b^2c^2d^2$; $16x^6-8x^4y^2+x^2y^4$.
(4) x^2-y^2; $4a^2-b^2$; $9-x^2$.
(5) $9a^2b^2-4b^4$; $16x^4-9y^4$; $a^6x^4-b^2y^8$.
(6) $36x^2y^2-25y^4$; $16x^{10}-1$; $1-9a^2b^6$.
(7) $a^4-8a^2x^2+16x^4$; $16b^4x^4-72b^2x^4y^2+81x^4y^4$.
(8) a^8-x^8. (9) $2a^2+2b^2$. (10) $4xy$.
(11) $-2a^2-16ab+4b^2$. (12) 0.
(13) $a^2-12ab+56b^2$. (14) $x^3-5x^2y+xy^2-5y^3$.
(15) $25a^4-25b^4$. (16) $24ab-120b^2$.

(17) $2x^3y^3+3x^2y^4-y^6$. (18) 0.
(19) $4a^2b^2-b^4$. (20) $4b^2$.

Ex. 13.

(1) $a+bc$; $3x-y$; $4a^2-3x^3$; $1+2y^2$.
(2) $5-4x$; $a^2b+2b^3c^2$; $7x^3-3yz^2$.
(3) $4a^2-4ab+b^2$, $4x^2-20xy+25y^2$.
(4) $\pm 2ax$; $\pm 4ay$; $\pm 10a^3b$; $\pm 12x^3y^2$.
(5) a^2; $4a^2$; z^2; $\frac{1}{4}$; $\frac{b^2}{4}$. (6) $4b^4$, $\pm 4bx^3$, $\frac{x^6}{16b^2}$.
(7) $(x+6y)(x-6y)$; $(2a+z)(2a-z)$;
$2(3y+5z)(3y-5z)$; $y(4y+ab)(4y-ab)$.
(8) $b^2(5ab+2c^2)(5ab-2c^2)$; $x^3y^2(6a^2+5bx)(6a^2-5bx)$;
$(10+xy)(10-xy)$.
(9) $(x+1)(x-1)$; $a^2(a+2)(a-2)$; $(1+3xz^2)(1-3xz^2)$;
$a(3ax^4+4)(3ax^4-4)$.
(10) $(a-b)(a+b)(a^2+b^2)$; $(2x-1)(2x+1)(4x^2+1)$;
$(a-1)(a+1)(a^2+1)(a^4+1)$;
$xy(x-y)(x+y)(x^2+y^2)$.
(11) $a+b$; $x-1$; $ax-2by$; $8a^2+5b^2$.
(12) $2x-y$; $3ax+1$; $4a^2-5xy$.

Ex. 14.

(1) x^2+5x+6; x^2+6x+5; $x^2-9x+18$.
(2) x^2-9x+8; x^2-7x-8; $x^2+3x-10$.
(3) $x^2+4x-21$; x^2-6x+8; $x^2+12x+11$.
(4) $x^2+ax-6a^2$; $x^2-(c+d)x+cd$; $x^2-3xy-4y^2$.
(5) $a^2-7ab+10b^2$; $x^4+3x^2y^2+2y^4$; $x^4-2x^3y-3x^2y^2$.

Ex. 15.

(1) $(x+3)(x+4)$; $(x-1)(x-6)$; $(x+3)(x+8)$;
$(x+1)(x+2)$.
(2) $(x-1)(x-4)$; $(x-1)(x+3)$; $(y+2)(y-5)$;
$(y+2)(y-3)$.
(3) $(x+1)(x+6)$; $(x-1)(x+7)$; $(x-y)(x-4y)$;
$(a-b)(a-2b)$.
(4) $5(x+1)(x-4)$; $3(y-5)(y+7)$; $(m+5a)(m-10a)$.
(5) $(x-a)(x+a)(x^2-3a^2)$; $2x^3(x-2)(x-6)$;
$(ax+6)(ax-9)$.
(6) $x-9$; $3ab-12$.

Ex. 16.

(1) $a^2+2ax+x^2+4a+4x+3$; x^2-4x-a^2+6a-5.
(2) $a^2-2ab+b^2-c^2$; x^4-x^2+2x-1.
(3) $a^2-2ab+b^2-c^2+2cd-d^2$; $x^2-2xy+y^2-z^2$.
(4) $4-x^2-2x^3-x^4$; $a^4+a^2b^2+b^4$.
(5) $(a-b+x)(a-b-x)$; $(x+2y+2z)(x+2y-2z)$; $(a-b+c+d)(a-b-c-d)$.
(6) $(x-y)(x+y+2)$; $(x+y)(x-y+2)$; $(a+x-2y)(a-x+2y)$.
(7) $(a-b+c)(a-b-c)$; $(a+b-c)(a-b+c)$; $(a+b+2)(a-b-2)$.
(8) $(x^2+xy+y^2)(x^2-xy+y^2)$; $(x^2+xy-y^2)(x^2-xy-y^2)$; $(a^2+a+1)(a^2-a+1)$.
(9) $a^2+2ab+2ac+b^2+2bc+c^2$; $x^2-2xy-2xz+y^2+2yz+z^2$; $x^4-4x^3+10x^2-12x+9$; $4x^6+4x^5-4x^4+x^2-2x+1$.
(11) $(a+x-1)(a+x-2)$; $(a-b)(a-c)$.
(12) $(x+y+2)(x+y-3)$; $(x^2+x+1)(x^2+x+3)$.

Ex. 17.

(1) $(x+b)(y-z)$; $(a-b)(b-c)$.
(2) $(x-y)(x+z)$; $(3x-y)(x-z)$.
(3) $(a+x)(a-x-b)$; $(a-x)(a-x+1)$.
(4) $(x-y)(3x+3y-2)$; $(m+n)(m-n)(p-q)$.
(5) $x(x+1)(x^2+1)$; $(ax-1)(a^3x^3-ax-1)$.
(6) $(3x-2y)(x-3y)(x+3y)$; $(2x-1)(2x^3+x^2+1)$.
(7) $(1-x)(1+x^2)$; $(1-x)^2(1+x)$.
(8) $(x+y+1)(x+y-xy-1)$.
(9) $2(a-d)(a+b+c+d)$.
(10) $(x+y-z)(x-y+z+1)$.
(11) $(a-3b)(a+2b-4)$.
(12) $(x-2y+1)(x-2)(x+2)$.

Ex. 18.

(1) $7x-4$. (2) $2a^2xy(3x-y)$.
(3) $2abc$. (4) $x-3$. (5) $2(a+b)$.
(6) $6x^2(x-y)$. (7) $3x(x+4)$. (8) $c(a-b)$.
(9) $5y(x-5y)$. (10) $x(x+1)$. (11) $3(x-1)$.
(12) $2(a-b)$.

Ex. 19.

(1) $5x-1$. (2) $x+1$. (3) $a(3a-1)$. (4) $9x^2-4$.
(5) $3x(3x^2-x+1)$. (6) $2(x-2)$.

Ex. 20.

(1) $2x-5$. (2) $a(a-2x)$. (3) $2x^2-3x+3$.
(4) $2x-7$. (5) $5x(6x-1)$. (6) $y(3x-4y)$.

Ex. 21.

(1) $3x-5$. (2) $4x+1$. (3) $3(x+3)$.
(4) x^2+xy+y^2. (5) $a-2$. (6) $2(2y+5)$.
(7) $3x(2x^2-5)$. (8) x^3-5x+1.
(9) $7x+1$. (10) $y(3x^2-xy-5y^2)$.

Ex. 22.

(1) $x-1$. (2) x^2+x+1. (3) $x+1$.
(4) $2(x+y)$. (5) x^2-xy+y^2. (6) $xy(x-y)$.

Ex. 23.

(1) $6x^2(x+1)$. (2) $24a^2c^2(a^2-c^2)$.
(3) $12x^2y^2(x^2-y^2)^2$. (4) $240(x^2-1)(x^2-4)$.
(5) $20a^2b(a-2b)(a-b)(a+3b)$.
(6) $(a-b)(b-c)(c-a)$. (7) $(a-b)(a-c)(b-c)$.
(8) a^4-b^4. (9) $x^2(x-1)(x-3)(x+4)$.
(10) $12xy(x-y)^3$.
(11) $(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$.
(12) $60xy(x^4-y^4)$. (13) $60(x-2)^2(x-3)$.
(14) x^8-y^8. (15) $30ab^2(a^2-4b^2)$.
(16) $(5x-1)(x+4)(2x+3)$.
(17) $2(3x-2y)(4x+3y)(3x+5y)$.
(18) $2x(x-1)(x^2-x-1)(x^2+1)$.
(19) $x^2(x^4-1)$.
(20) $12(x-3)(4x^2-1)(3x+2)$.

Ex. 24.

(1) $2x+3+\dfrac{10}{x-4}$. (2) $a+b+\dfrac{2b^2}{a-b}$.

(3) $x-1+\frac{5x+4}{5x^2+4x-1}$. (4) $\frac{a^2}{a+1}$.

(5) $\frac{x^2}{x-3}$. (6) $\frac{2a^2-ab-b^2}{a+b}$. (7) $\frac{x-1}{4x}$.

(8) $\frac{a+b}{a-2b}$. (9) $\frac{3(a+b)}{4ab(a-b)}$. (10) $\frac{2a^2}{a^2-b^2}$.

(11) $\frac{a^3-a^2b-ab^2-b^3}{ab(a^2-b^2)}$. (12) $\frac{3(a-x)}{2x}$.

(13) $\frac{x(3x-y)}{y(3x+2y)}$. (14) $\frac{a}{a+4}$. (15) $\frac{a+b+c}{a}$.

(16) $\frac{3(x-y)}{x^2y^2}$. (17) $\frac{3b(a+2b)}{2a(a+3b)}$.

(18) $\frac{ab+ac+bc-a^2-b^2-c^2}{(a-b)(b-c)(c-a)}$.

(19) $\frac{4b^2-3a^2}{ab(a+2b)}$. (20) $\frac{7x-17}{4x(x-1)(x-2)}$.

(21) $\frac{ax}{(a-x)(a+2x)}$. (22) $\frac{9a-10b+c}{(a-b)(b-c)(a-c)}$.

(23) $\frac{-4}{(x-1)(x-2)(x-3)}$. (24) $\frac{-2(x^2-xy+y^2)}{xy(x-y)}$.

(25) $\frac{2(b-a)}{(a+b)y}$. (26) $\frac{2x^3-5x^2y+10xy^2+5y^3}{(x^2-y^2)^2}$.

(27) $\frac{2(a^3-a^2x+ax^2-x^3)}{ax^2}$. (28) 1.

(29) $\frac{-x}{x^4+x^2+1}$.

(30) $\frac{(b-c)(b+c-a)}{(a+b+c)(a+b-c)(a-b+c)}$.

(31) $\frac{22x^2-34x-3}{6x(x^2-1)(x-4)}$. (32) $\frac{a^4-4ax^3-x^4}{a^4-x^4}$.

(33) 0. (34) 0.

(35) $\frac{xy^2}{(x-y)^2(x+y)}$. (36) $\frac{a+b+c+d}{a-b+c+d}$.

(37) $\frac{3ab(x-y)}{a^2x^2-b^2y^2}$.

(38) $\frac{a+b+c}{(a-b-c)(a-b+c)(a+b-c)}$.

(39) $\frac{2x^5y^5}{x^5-y^5}$. (40) $\frac{1}{(x+1)^2}$. (41) $\frac{x^2+1}{2x}$.

(42) $\frac{x^2y}{2}$. (43) $\frac{ax}{x+a}$. (44) $\frac{z}{(x-z)(y-z)}$.

(45) $\frac{z(x+y-z)}{x(y+z-x)}$. (46) $\frac{1}{z(x-z)(z-y)}$.

(47) $\frac{2a}{a^2+ab+b^2}$. (48) $\frac{ax+by}{ax-by}$.

(50) a. (49) $\frac{(x+y)(y+z)(z+x)}{x+y+z}$.

Ex. 25. $x=$

(1) 4. (2) 18. (3) -3. (4) 3.

(5) $\frac{1}{2}$. (6) 36. (7) 10. (8) 37.

(9) $-4\frac{8}{13}$. (10) 6. (11) 8. (12) 9.

(13) $-\frac{11}{12}$. (14) -1. (15) $\frac{157}{211}$. (16) $\frac{4}{5}$.

(17) 7. (18) -5. (19) $-\cdot 5225$. (20) 7.

(21) 4. (22) $\frac{3}{5}$. (23) $\frac{abc}{a+b}$. (24) a^2+b^2.

(25) $\frac{1}{2a+1}$. (26) $\frac{6}{5}$. (27) 5. (28) $-\frac{23}{13}$.

(29) $-\frac{1}{4}$. (30) $\frac{2c}{a-b}$. (31) $\frac{acd+b^2d+bc^2}{bcd^2}$.

(32) 3. (33) $\frac{b}{c}$. (34) 2. (35) 5.

(36) -9. (37) $\frac{a}{4}$. (38) 0. (39) $-\cdot 3\dot{6}$.

(40) 0. (41) $\frac{15}{26}$. (42) $-\frac{adf+bcf+bde}{(g+h)bdf}$.

(43) $\frac{2ab}{a+b}$. (44) $\frac{ab-ac-bc}{c^2}$.

(45) $\frac{(a-b)^2}{3(a+b)}$. (46) -1.

(47) $-\frac{a^2+3ab+b^2+c^2}{a+b}$. (48) $\frac{(a-b)(a-c)}{(a+b)(a+c)}a$.

(49) $\frac{c^2+2ac-a^2}{2c}$. (50) 0.

Ex. 26.

(1) 13, 4. (2) 42s., 36s. (3) 55, 24.
(4) 62, 63, 64, 65. (5) 175 miles. (6) 13 ft., 8 ft.
(7) 11s., 13s., 19s. (8) £3 10s.; 10s. (9) 30, 5.
(10) $56\frac{1}{4}$ ft. (11) 60.
(12) 3 miles, 5 miles.
(13) 6s. 6*d*., 5s. 6*d*., 4s. 6*d*., 3s. 6*d*.
(14) 6s. 10*d*.; 13s. 2*d*. (15) 32.
(16) 37. (17) 36 gallons. (18) £15.
(19) £69, £32. (20) 378 inches. (21) £18 12s.
(22) 18. (23) 10. (24) 4. (25) 63.
(26) 12, 20, 4, 64. (27) 70, 35, $17\frac{1}{2}$ hrs.
(28) 2, 4, $1\frac{1}{3}$ hrs. (29) 30, 15 miles. (30) 10 hrs. 40 m.
(31) 94. (32) 111, 69 lbs.
(33) $\frac{a^2+b^2}{2a}, \frac{a^2-b^2}{2a}$. (34) £$\frac{ma+b}{m+1}$.
(35) $\frac{ab}{a-b}$. (36) 37, 38, 39.
(37) 126 miles. (38) 32, 40.
(39) 12,000 sq. yds.; 2s. 3*d*. (40) 1320 yds.

Ex. 27.

(1) $x=2, y=1$. (2) $x=2, y=-1$.
(3) $x=2, y=8$. (4) $x=29, y=23$.
(5) $x=\frac{3}{2}, y=-1$. (6) $x=6, y=3$.
(7) $x=9, y=8$. (8) $x=-3, y=-2$.
(9) $x=1, y=8$. (10) $x=-76, y=-26$.
(11) $x=-1, y=5$. (12) $x=\frac{bd-c^2}{a(b-c)}, y=\frac{c-d}{b-c}$.
(13) $x=\frac{abc}{b-a}, y=\frac{abc}{a-b}$.
(14) $x=\frac{a^2c+ab-abc^2}{ac+b}, y=\frac{b^2}{ac+b}$.
(15) $x=\frac{1}{2}, y=1$. (16) $x=1\frac{2}{3}, y=0$.
(17) $x=b-a, y=a-b$. (18) $x=-1, y=-\frac{1}{2}$.
(19) $x=\frac{(a+b)ab}{(a^2+ab+b^2)c}$; $y=-\frac{a+b}{c}$.
(20) $x=\frac{ab(a+b)}{2(a^2+b^2)}$; $y=\frac{a^3-b^3}{2(a^2+b^2)}$.

Ex. 28.

(1) $x=1, y=2, z=-3$. (2) $x=1, y=-1, z=2$.
(3) $x=-1\frac{1}{2}, y=2\frac{1}{2}, z=6\frac{1}{2}$.
(4) $x=-4, y=-3\frac{2}{3}, z=-4\frac{1}{2}$.
(5) $x=6, y=-12, z=18$. (6) $x=\frac{1}{3}, y=\frac{1}{4}, z=\frac{1}{6}$.
(7) $x=\dfrac{a+b}{2a}, y=\dfrac{ab+ac-b^2+bc-2c^2}{2(b^2-c^2)}$,
$$z=\frac{3bc-ab-ac-b^2}{2(b^2-c^2)}.$$
(8) $x=1, y=\frac{1}{2}, z=\frac{1}{3}$.
(9) $x=0, y=-1, z=2, v=-4$.
(10) $x=-1, y=-2, z=-3, u=-4, v=-5$.
(11) $x=17, y=22, z=-25$.
(12) $x=-(a+b), y=\dfrac{(a+b)(a-b-c)}{a+b+c}$
$$z=\frac{(a+b)(a+b-c)}{a+b+c}.$$

Ex. 29.

(1) 8 florins, 11 halfcrowns. (2) A, 31*s.*; B, 27*s.*
(3) $\frac{6}{18}$. (4) 4*s.* 4*d.*; 1*s.* 7*d.*
(5) 48, 28. (6) $7\frac{1}{2}$ tons, 14 tons. (7) 3 m., 4 m.
(8) 69. (9) $\frac{2}{3}, \frac{5}{6}$. (10) 2660 yds.
(11) 216 m., 5 hrs. 24 min. (12) 9*s.*, 4*s.* 6*d.*
(13) $\frac{24}{36}$. (14) 8 m., 10 m., per hour.
(15) 54, 36. (16) 717.
(17) 36 days, 45 days. (18) 6*s.* 3*d.*, 5*s.*
(19) 45 turkeys, 60 geese. (20) 7.
(21) A, £10; B, £6; C, £4. (22) 360, 72 yds.; 20, 12 yds.
(23) 200, 160 yds. (24) £25 4*s.*
(25) 16, 20, 42. (26) 315 miles.
(27) A, 10*s.*; B, 5*s.*; C, 2*s.* (28) 30 miles per hour.
(29) 36, 18, 10 galls. (30) 9, $8\frac{2}{11}$.

Ex. 30.

(1) 3292. (2) $2x^3y-x^2y^2+3xy^3$.
(3) $4a^5-19a^4b+38a^2b^3+5ab^4$.
(4) $4a^2+2ab+b^2$. (5) $x+\frac{12y}{a}$.
(6) a, a^2, b, ab; x, x^2, y, xy, x^2y.
(7) $7a-6x$; $ab+ac-bc$; $-\frac{ab^2+2ac^2}{bc}$.
(8) $(x+2y)(x-2y)$; $3xy(x+y)(x-y)$; $(x+2)(x-9)$.
(9) 3; -2; $-\frac{2}{5}$; $7, 6$. (10) x^2+x+1.
(11) 36. (12) $55, -1\frac{4}{9}$; $1\frac{1}{18}, 2\frac{67}{72}$.
(13) $-2m^3+m^2n+3mn^2+7n^3$.
(14) $8a^5b-26a^4b^2+16a^3b^3+14a^2b^4-12ab^5$.
(15) $3x^3+x^2y-xy^2$. (16) $60a^4b^3c^6$; $8a^3b^3xy^2$.
(17) $\frac{3}{a}$; $\frac{3xy^3}{10z^3}$. (18) $\frac{20bn}{c}$.
(19) $1\frac{1}{10}$; $1\frac{3}{4}$. (20) $-2a^3-3x^3$.
(21) $a^4-a^3b-2a^2b^2+8ab^3+2b^4$.
(22) $8x^5y-26x^3y^3+2xy^5$. (23) $3x^3+2x^2y-y^3$.
(24) $7a^2x^3y$, $140a^3bx^4y^2$. (25) $\frac{x^3+4x^2y-y^3}{x^2y^2}$.
(26) $a-\frac{bc}{20}+\frac{xy}{240}$; $\frac{20a-bc}{21}+\frac{xy}{252}$.
(27) 5. (28) $5x^4+4x^3+3x^2+2x+1$.
(29) $\frac{x^2-5x+5}{x^2-x+1}$. (30) 5; -107.
(31) 28; $4x^2y+7xy^2+y^3$.
(32) $3x-7$; $60x^2y^2(x+y)^2(x-y)$.
(33) $\frac{-a^3-a^2b-3ab^2+b^3}{ab(a^2-b^2)}$; $\frac{4}{3(x+1)}$.
(34) $2\frac{3}{13}$; $-\frac{1}{2}, -\frac{1}{4}$; 3, 4, 1.
(35) $\frac{5}{8}$. (36) $5\frac{5}{11}'$ and $16\frac{4}{11}'$ past 2.
(37) $4a^2b(a-b)$; $4ab(a+b)(a-b)$; $12ab(a-b)(a+3b)$.
(38) $60x^3y^5z^4$. (39) $1\frac{2}{9}$.
(41) $a^4b+a^3b^2-a^2b^3-ab^4$; a^3, a^2b, ab^2, b^3.
(42) 20; 279; 107.
(43) $a^4-a^2b^2+b^4$; $a^4-6a^3b-3a^2b^2+2ab^3-b^4$.
(44) $5a^3x^5-21a^4x^4+2a^5x^3-11a^6x^2-3a^7x$.
(45) $3x^3y-5x^2y^2-xy^3$. (46) $\frac{-6x^3+25x^2y-4y^3}{12xy^2}$.

(47) $\frac{ac+bc}{ab}$. (48) $-\frac{ab}{c}$. (49) 13, 5″, 1′.

(50) $(x+y-z-1)(x-y-z+1)$. (51) 0.

(52) $\frac{20a}{b}$. (53) $5a^3-3a^2c+5ac^2$.

(54) $28x^4y^2-50x^3y^3+18xy^5$; $6a-2b$.

(55) 0; $210a-8b$. (56) $\{(a+b)-(c-d)\}(e^2+f^2)$.

(57) $-3a$.

(58) $12(x-3)$; $(x+3)(x-3)$; $(x+2)(x+3)$; $(x-2)(x+3)$; $12(x^2-4)(x^2-9)$.

(59) $2(2a-b)$. (60) $\frac{xy+z}{xy+y+z}$; $\frac{a^2+a-1}{a^2+2a-2}$.

(61) $\frac{19mb}{a}$. (62) $a+b+c$.

(63) $a^3-a^2b+ab^2-b^3+a^2c-b^2c$.

(64) $4x^2+y^2$. (65) $6a^2c$, $x(3x+y)$.

(66) $\frac{2b}{(a-b)(b-c)}$; $\frac{x^3+2x^2y+2xy^2+y^3}{x^2y^2(x-y)}$.

(67) 120; $\frac{1}{2}$; 4, -1. (68) £760. (70) $11a-9b$.

(71) 4. (72) x^6-a^6. (73) bx^2+cx-e.

(74) $x(x-2)$. (75) 2; $\frac{9}{10}$.

(76) $\frac{8}{15}$; $-\frac{6}{11}$; $\frac{9}{13}$; $\frac{5}{2}$, $\frac{1}{2}$. (77) $\frac{a}{4}$, $\frac{3a}{4}$.

(78) 22 crowns, 33*s*. (79) $\frac{23c}{8}$.

(80) x^2+2x+1. (81) $5a-b$; $11-x^2$.

(82) $(x+1)(x-3)$; $(x+1)(x-1)$; $2x(x-3)$; $2x(x^2-1)(x-3)$. (83) $2a-b$.

(84) $\frac{x+y+z}{x-y+z}$; $\frac{2}{(x-1)(x-2)(x-3)}$.

(85) 14; $-\frac{ab}{a+b}$; 2, -1, 2. (86) 132.

(88) $-\cdot016$. (89) 12 miles.

(90) $-2xy(x+y)$; $5(x-y)(x-3y)$.

(91) $ab(4a-b)$; $6a^2b^2(4a-b)(3a-2b)(5a+3b)$.

(92) $\frac{p-2q+r}{q(p+r)}$; $\frac{4q}{(p-q)(p-3q)}$.

(93) $\frac{3}{2}$; $-\frac{5}{31}$, $-\frac{23}{31}$. (94) 7. (95) 22.

(97) $p+2q+r$.

(99) $x=y=z=\frac{1}{a+b+c}$. (100) $x+\frac{b-q}{a-p}$.

Ex. 31.

(1) a^6, x^{10}, x^4y^6, $4a^4b^2c^6$, $25a^2x^6y^4$, $49m^6n^2x^4y^8$;
a^9, x^{15}, x^6y^9, $8a^6b^3c^9$, $-125a^3x^9y^6$, $343m^9n^3x^6y^{12}$.

(2) $32a^{10}b^5$, $-243x^5y^{15}$, $-3125a^{10}b^5x^{15}$, $\frac{a^{15}b^{10}}{32}$, $-\frac{32x^{15}y^5}{243a^5b^5c^5}$;
$64a^{12}b^6$, $729x^6y^{18}$, $15625a^{12}b^6x^{18}$, $\frac{a^{18}b^{12}}{64}$, $\frac{64x^{18}y^6}{729a^6b^6c^6}$.

(3) $a^3+3a^2x+3ax^2+x^3$; $16x^4-32x^3y+24x^2y^2-8xy^3+y^4$;
$x^{12}y^6-12x^{11}y^7+60x^{10}y^8-160x^9y^9+240x^8y^{10}-192x^7y^{11}$
$+64x^6y^{12}$;
$a^7b^7-21a^6b^6+189a^5b^5-945a^4b^4+2835a^3b^3-5103a^2b^2$
$+5103ab-2187$.

(4) $1-2a-a^2+2a^3+a^4$;
$8-36x+102x^2-171x^3+204x^4-144x^5+64x^6$;
$a^{10}+5a^9y+15a^8y^2+30a^7y^3+45a^6y^4+51a^5y^5+45a^4y^6$
$+30a^3y^7+15a^2y^8+5ay^9+y^{10}$.

Ex. 32.

(1) a^2, x^4, $2a^3b$, $3x^2y^3$, $7ax^3y^5$, $10mn^2x^3y^5z^4$.
(2) 4, x^2, $2a^3bc^2$, $12c^3d^2xy^4$. (3) $2ab^2c$.
(4) $36x^3y^3z^3$. (5) 92. (6) 40.

Ex. 33.

(1) $3a+2b$. (2) $5ab-2bc$. (3) a^2+2a-1.
(4) x^2-xy+y^2. (5) $2a^3-3a^2x-ax^2$.
(6) $4p^2-4pq+q^2$. (7) $3x^3-4xy^2-2y^3$.
(8) $2a^4+4a^2c^2-4c^4$. (9) $m^4-2m^3+3m^2-4m+5$.
(10) $5x^3-3x^2y-4xy^2+y^3$. (11) $a^3+3a^2b+3ab^2+b^3$.
(12) $x^2-\frac{1}{2}xy-y^2$. (13) $x^2-2xy+y^2-\frac{y^3}{x}$.
(14) $\frac{a^2}{3}-\frac{3ax}{4}+\frac{x^2}{2}$. (15) $1+\frac{2}{x}+\frac{3}{x^2}+\frac{4}{x^3}$.
(16) $x^2-(a-b)x+ab$. (17) $\frac{a}{b}-1+\frac{b}{a}$.
(18) $1+\frac{a}{2}-\frac{a^2}{8}+\frac{a^3}{16}-\frac{5a^4}{128}$. (19) $2+\frac{x^2}{2}-\frac{x^4}{16}+\frac{x^6}{64}$.
(20) $10a^2+2ab+b^2$.

Ex. 34.

(1) $a+2b$. (2) $2a+x$. (3) $xy+4y^2$.
(4) $2b-5y$. (5) $3a^2+2bc$. (6) x^2+xy+y^2.

(7) a^2-2a-1. (8) $4x^2+4x-1$.
(9) $1-3x+4x^2$. (10) $x^2+3xy-9y^2$.
(11) $1-x+x^2-x^3$. (12) $1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}$.

Ex. 35.

(1) $3a^2-5b^2$. (2) $2-x$. (3) $1-y^2$.

Ex. 36.

(1) ± 5. (2) ± 3. (3) ± 2. (4) ± 1.
(5) $2, -6$. (6) $8, -2$. (7) $8, -1$. (8) $\frac{1}{2}, 11\frac{1}{2}$.
(9) $3, -2$. (10) $\frac{1}{2}, \frac{1}{6}$. (11) $2\frac{1}{3}, -1$. (12) $6, -1\frac{1}{3}$.
(13) $1, -\frac{2}{5}$. (14) $\frac{1}{4}, -\frac{1}{3}$. (15) $14, -\frac{1}{2}$. (16) $0, 1\frac{3}{5}$.
(17) $2, \frac{8}{9}$. (18) $4, -2$. (19) $0, 11$.
(20) $-3, -3\frac{8}{9}$. (21) $0, -a$. (22) $0, \frac{b}{a}$.
(23) $0, 3$. (24) $5, -1$. (25) $5, 1\frac{12}{17}$.
(26) $10, -8\frac{2}{3}$. (27) $\cdot 1, \cdot 25$. (28) $\frac{1}{2}, -4\frac{3}{5}$.
(29) $\frac{a}{4}, \frac{3a}{4}$. (30) $a-b, b-3a$. (31) $\pm\sqrt{10}$.
(32) $\pm\sqrt{2}$. (33) $3, -1$. (34) $1\pm\sqrt{3}$.
(35) $3, -6$. (36) $0, -1\frac{10}{13}$. (37) $\pm\sqrt{3}$.
(38) $\frac{a+b}{3}, \frac{b-a}{2}$. (39) $-a, -b$. (40) $a+b$.
(41) $0, \pm 6$. (42) $1, 2, 3$. (43) $0, -1\pm\sqrt{6}$.
(44) $\pm 1, \pm 2$. (45) $1, 2, 4, -3$.
(46) $1, -1$. (47) $0, a, \pm b$.
(48) $a-2, -\frac{1}{2}\{a+1\pm\sqrt{-3a^2+6a+1}\}$.
(49) $\frac{1}{2}\{1\pm\sqrt{1-4a}\}$. If $a=\frac{1}{4}$.
(50) $\frac{1}{2}\{p\pm\sqrt{p^2-4q}\}$. $q=2\frac{1}{4}$; $p=\pm 6$.

Ex. 37.

(1) $\pm 1, \pm 2$. (2) $\pm 3, \pm\frac{1}{2}$. (3) $\frac{1}{2}, -2$.
(4) $\pm 1, \pm\frac{1}{2}$. (5) $0, \frac{1}{3}, -\frac{1}{2}$. (6) $1, \frac{1}{2}$.
(7) $1, -3, 5$. (8) $2, 3, \frac{1}{2}, -\frac{1}{2}$.
(9) $0, 1, 2, 3, 4$. (10) $\frac{1}{2}, \pm\frac{1}{3}, 1\frac{1}{6}$.

Ex. 38.

(1) 4 or -5. (2) ± 6. (3) 20 ft., 15 ft.
(4) 84, 16. (5) 12 in. (6) 3, 4, 5.

(7) 16 in., 16 in., 28 in. (8) 8 or $-3\frac{3}{7}$. (9) $1\frac{1}{2}$ yds.
(10) 36 hrs., 45 hrs. (11) 60 yds. at 5*s.*; or 6 yds. at 6*d.*
(12) 8 ft., $4\frac{1}{2}$ lbs. (13) 15, 36, 39 ft.
(14) 16*d.* (15) 20, 15. (16) 39.
(17) 15, 9, 6 yds.; or 80, 48, 32 in.
(18) 56. (19) 6 ft. (20) 4 ft.
(21) 3*s.*, 9*s.* 6*d.* (22) $2\frac{2}{3}$ yds.
(23) $20\frac{5}{11}$, $24\frac{6}{11}$ miles. (24) 12·360 . . 7·639 . . in.
(25) 14 acres at £75. (26) 10″.
(27) 56*s.*, 35*s.* (28) Rad. = 2·414 . . in.
(29) 40, 30. (30) $5\frac{5}{8}$ m. per hour.

Ex. 39. (*Impossible* roots are omitted.)

(1) $x=3, y=2$; or $x=12, y=-7$.
(2) $x=4, y=1$; or $x=-2\frac{3}{4}, y=-1\frac{1}{4}$.
(3) $4, -3$; or $-3, 4$. (4) $4, -3$; or $3, -4$.
(5) $\pm 5, \pm 2$; or $\pm\frac{7}{\sqrt{2}}, \pm\frac{3}{\sqrt{2}}$.
(6) $8, \frac{3}{2}$; or $-3, -4$. (7) $\pm 1, \pm 7$; or $\pm 7, \pm 1$.
(8) $2, -1$; or $-4\frac{18}{19}, -2\frac{15}{19}$. (9) $8, -1$; or $1, -8$.
(10) $2\frac{1}{2}, 1\frac{1}{2}$; or $-1\frac{1}{2}, -2\frac{1}{2}$. (11) $3, 4$; or $-3\frac{1}{7}, -3\frac{19}{28}$.
(12) $\pm 1, \pm\frac{1}{2}$. (13) $\pm\frac{4}{5}, \pm\frac{4}{7}$.
(14) $4, -4$. (15) $3, 2$; or $-2, -3$.
(16) $\pm 2, \mp 1$; or $\pm 1, \mp 2$.
(17) $\pm\frac{2}{3}, \mp\frac{1}{3}$; or $\pm\frac{7}{\sqrt{171}}, \pm\frac{10}{\sqrt{\ [illegible]}}$.
(18) $\frac{a\pm 2b}{2}, \frac{a\mp 2b}{2}$. (19) $\pm\frac{a^2+b^2}{a-b}, \pm\frac{2ab}{a-b}$.
(20) $\pm(a+b), \pm(a-b)$. (21) $\frac{1}{2}, \frac{1}{3}$; or $\frac{1}{3}, \frac{1}{2}$.
(22) $6\frac{1}{2}, 1\frac{1}{2}$; or $\frac{39}{46}, -\frac{9}{46}$. (23) $2, -1$; or $1, -2$.
(24) $\pm\frac{3}{2}, \pm\frac{3}{2}$; $\pm\frac{1}{2}\sqrt{3}, \mp\frac{1}{2}\sqrt{3}$.
(25) $0, \pm 4$; or $\pm 4, 0$. (26) $0, -a$; or $a, 0$.
(27) $6, 4$; or $-3\frac{4}{5}, -3\frac{7}{20}$. (28) $\pm\frac{a+b}{a}, \mp\frac{a+b}{b}$.
(29) $\pm\frac{1}{5}(2a-3b), \pm\frac{1}{5}(2b-3a)$.
(30) $x, -2x$; or $-2x, x$.

$$-\frac{1}{2x^2}\left\{a^3 \pm \sqrt{a^6 - 4a^3x^3}\right\},\quad -\frac{1}{2x^2}\left\{a^3 \mp \sqrt{a^6 - 4a^3x^3}\right\}.$$

Ex. 40.

(1) 7, 5. (2) 12, 16. (3) $\frac{2}{3}$, $\frac{3}{4}$. (4) 8, 10 ft.
(5) 37, 73. (6) 88, 55 yds. (7) 4 miles.
(8) 20. (9) 120 yds. (10) 390 yds.

Ex. 41.

(1) 8, 3; or 19, 1. (2) 1, 11; or 4, 4.
(3) 5, 4. (4) 19, 1; or 9, 2.
(5) 7, 5; or 15, 2. (6) 4, 13; or 9, 5.
(7) 6, 2. (8) 8, 15; or 18, 8; or 28, 1.
(9) $x = 5, 19, 33, \ldots, y = 1, 6, 11, \ldots$
(10) $x = 3, 10, 17, \ldots, y = 5, 17, 29, \ldots$
(11) 8, 1. (12) 21, 12.
(13) Two. (14) 6, 17; or 13, 7.
(15) 3*s.* 2*d.*, 1*s.* 10*d.*; or 3*s.* 7*d.*, 11*d.*
(16) $3\frac{3}{4}$, $1\frac{7}{8}$; $4\frac{2}{7}$, $1\frac{3}{7}$; $4\frac{8}{13}$, $1\frac{2}{13}$; &c.
(17) 3, −2; 1, 1; −1, 4; −3, 7; &c.
(18) $\frac{1}{4}$, $1\frac{3}{4}$. (19) 1, 4; 4, $2\frac{1}{3}$; 7, $\frac{2}{3}$
(20) 4, 5, 1; 5, 3, 2; or 6, 1, 3.
(21) 2, 1, 2; or 2, 3, 1.
(22) 24, 14, 32; 18, 28, 24; or 12, 42, 16.
(23) 10, 14, 176. (24) 149.
(26) 34, 13, 1; 37, 8, 3; or 40, 3, 5.
(27) 905, 3620 sq. yds.
(28) 24 in., 4 in.; or 12 in., 16 in.
(29) Larger 34, $12\frac{3}{4}$, $9\frac{5}{7}$, $8\frac{1}{2}$, ... feet,
Smaller 17, $4\frac{1}{4}$, $2\frac{3}{7}$, $1\frac{7}{10}$, ... feet.
(30) (i) 21, 33. (ii) 41, 1. (iii) 1, 65.

Ex. 42.

(1) 24 : 35; first; second. (2) 10 : 9.
(4) 15, 25. (5) 6*s.*, 7*s.* 6*d.*, 18*s.*
(12) 3 or $\frac{4}{7}$. (14) 0 or $\frac{1}{2}$; ±2.
(16) 70. (17) 3 : 7; 45 : 104.
(18) 7 : 5. (20) 25 : 34.

Ex. 43.

(1) $A = \frac{4}{5}B$, $9\frac{3}{5}$. (2) $\frac{2}{9}$. (3) 8. (4) 15.
(5) $161\frac{1}{9}$ ft.; $402\frac{7}{9}$ ft. (6) $9\frac{1}{7}$ ft.
(7) $20\frac{4}{7}$ ft. (8) £520.

(9) 32 lbs. 4·2 oz. (10) 20 lbs. 10·368 oz.

(11) ·2385 . . . (13) $\frac{ab}{a+b}\left(\frac{b-a}{y}+\frac{2ab}{y^2}\right)$.

(14) 1 inch. (15) 357·5 : 1 nearly.

Ex. 44.

(1) 53; −6; 13; −29; $a+21b$; $a+7b$.

(2) $7\frac{1}{2}$; 6; a^2. (3) 25th. (4) −7.

(5) $7\frac{1}{3}$. (6) $5\frac{1}{2}$, 10, $14\frac{1}{2}$; 1, 5, 9, 13.

(7) $r(x-y)-\frac{1}{2}(x-5y)$.

(8) 185; −48; 35; −93; $\frac{(3n-1)na}{2}$; $11(a+9b)$.

(9) 1, $2\frac{2}{5}$, $3\frac{4}{5}$, $5\frac{1}{5}$, $6\frac{3}{5}$, 8. (10) 9.

(11) −10. (12) 6, 14. (13) 6.

(14) 3, 5, 7. (15) 8. (16) 14.

(17) 10. (19) 3, 6, 9, 12, 15. (20) 10″, $\frac{1}{4}$.

Ex. 45.

(1) 1458; 96; $\frac{3}{128}$; −128; x^{14}; $\frac{81}{4}m^4a^5$.

(2) 10; 1; $9a^3b^4$; $30x^2y^2z$. (3) 3, 192.

(4) 4. (5) 20, 50; 28, 56, 112.

(6) 5th. (7) $\frac{128}{27}x^7$.

(8) 765; −182, 547; $15\frac{63}{64}$; 1953·1; $\frac{205}{256}m$; $\frac{x^9-y^9}{x^4(x-y)}$.

(9) 3. (10) $1\frac{127}{128}$. (12) $\frac{1}{2}$, 1, 2, 4.

(13) 12, 18, 27; or 12, −6, 3.

(15) $\frac{32}{243}$; $\left(\frac{m-1}{m}\right)^n$. (16) ·59049 of whole.

(17) $\frac{(a-b)^n}{a^{n-1}}$; $\frac{(a-b)^n-a^n+na^{n-1}b}{a^{n-1}}$.

(18) 1, $2\frac{1}{4}$. (19) 6.

Ex. 46.

(1) $4\frac{1}{2}$; $\frac{2}{3}$; 6; $\frac{1}{6}$; $3\frac{1}{3}$. (2) $\frac{1}{4}$.

(3) 3, $2\frac{2}{5}$, . . . or 27, $-21\frac{3}{5}$, . . .

Ex. 47.

(1) 12; $7\frac{1}{7}$; $\dfrac{72}{10-n}$; $\dfrac{ab}{(n-1)a-(n-2)b}$.

(2) $13\frac{1}{3}$, $14\frac{2}{17}$, ... $17\frac{1}{7}$, $18\frac{6}{13}$.

(3) $12\frac{6}{7}$, $13\frac{11}{13}$, 15, $16\frac{4}{11}$, 18; $\dfrac{(m+1)ab}{a+mb}$, $\dfrac{(m+1)ab}{2a+(m-1)b}$, . .

(5) Sixth. (6) 6, $6\frac{2}{5}$, $6\frac{6}{7}$, $7\frac{5}{13}$.

(8) 5, 7. (9) $\dfrac{ab-2ac+bc}{a-2b+c}$.

Ex. 48.

(1) 2300, $\dfrac{n(4n^2-1)}{3}$. (2) 3805.

(3) 8128. (4) $\dfrac{x-11x^{11}+10x^{12}}{(1-x)^2}$.

(5) $\dfrac{1636}{2187}$, $\dfrac{3}{4}$; $\dfrac{412}{2187}$, $\dfrac{3}{16}$.

Ex. 49.

(2) 6. (3) 120. (4) 12.

(5) 4845, 969, 3876. (6) 95040.

(7) 5040, 144. (8) 31824.

(9) 720, 1956. (10) 60.

(11) 720, 90720, 45360, 129729600.

(12) 6160. (13) 161700.

(14) 1368000. (15) 56; 3268.

(16) 120; 24 times; 3999960. (17) 15.

(18) $\dfrac{m(m-1)\ .\ .\ .\ (m-r+1)}{\underline{|r}} \cdot \dfrac{n(n-1)\ .\ .\ .\ (n-s+1)}{\underline{|s}}$

or $\dfrac{\underline{|m} \cdot \underline{|n}}{\underline{|r} \cdot \underline{|m-r} \cdot \underline{|s} \cdot \underline{|n-s}}$.

(19) 1296. (20) 181440. (21) 17.

(22) 186648; 924. (23) 11, 3. (24) 5775.

Ex. 50.

(1) 35; -15. (2) $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

(3) $1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$.

(4) $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$.

(5) $1+10x+40x^2+80x^3+80x^4+32x^5$.

(6) $x^8-24x^7+252x^6-1512x^5+5670x^4-13608x^3$
$+20412x^2-17496x+6561$.

(7) $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$.

(8) $1-3x+4x^2-\frac{28}{9}x^3+\frac{14}{9}x^4-\frac{14}{27}x^5+\frac{28}{243}x^6-\frac{4}{243}x^7$
$+\frac{1}{729}x^8-\frac{1}{19683}x^9$.

(9) $1-\frac{15}{4}y+\frac{45}{8}y^2-\frac{135}{32}y^3+\frac{405}{256}y^4-\frac{243}{1024}y^5$.

(10) $190a^{18}x^2$.
(11) $-14080000x^9y^3$.

(12) $\frac{35}{1944}x^4y^6$.
(13) $-1365a^{19}x^{11}$.

(14) $-115200x^{11}y^{16}$.
(15) 70.

(16) -252.
(17) $-35\left(\frac{x}{y}-\frac{y}{x}\right)$.

(18) $\frac{n(n-1)\ .\ .\ .\ (n-r+2)}{1\ .\ 2\ .\ .\ .\ .\ (r-1)}(2a)^{n-r+1}x^{r-1}$;
$\frac{n(n-1)\ .\ .\ .\ (n-r+2)}{1\ .\ 2\ .\ .\ .\ .\ (r-1)}(2a)^{r-1}x^{n-r+1}$.

(19) $(-1)\ \frac{n(n-1)\ .\ .\ .\ (n-r-2)}{1\ .\ 2\ .\ .\ .\ .\ (r+3)}a^{n-r-3}x^{r+3}$.

(20) $\frac{2n(2n-1)\ .\ .\ .\ (n+1)}{1\ .\ 2\ .\ .\ .\ .\ .\ .\ n}a^nx^n$, or $\frac{\underline{|2n}}{(\underline{|n})^2}a^nx^n$.

(21) $(-1)^n\frac{\underline{|2n+1}}{\underline{|n}\ \underline{|n+1}}(a^{n+1}x^n-a\,x^{n+1})$.

(22) 9.
(23) 4096, 531441; 1, 0.

(24) 1, 2, 11.

Ex. 51.

(1) $2r^4+3r^3+8r^2+7r+6$; $6r^5+7r^4+2r$.

(2) $2r^4+7r^3+20r^2+23r+20$.

(3) 530333, 10044042; 123621, 1056472.

(4) 964941.

(5) $2(4^4)+(4^3)+(4^2)+1$; $(9^4)+4(9^3)+6(9^2)+4(9)+1$.

(6) 6.
(8) 34111110; 112022.

(9) 23861. (10) 9999, 1000; r^4-1, r^3; r^n-1, r^{n-1}.
(11) $p-q$ or $p-q+1$. (12) 7. (13) 9.
(14) $2p$ or $2p-1$; $\frac{p}{2}$ or $\frac{p+1}{2}$. (15) 1111.
(16) $2(8^4)+5(8^3)+7(8^2)$. (17) $\cdot 22$, $\cdot 01$.
(18) $20\cdot2444$, $\cdot\dot{4}\dot{0}$. (19) 3.
(20) $3p$, $3p-1$, or $3p-2$; $\frac{x}{3}$, $\frac{x+1}{3}$, $\frac{x+2}{3}$.

Ex. 52.

(1) $5x^2-3ax+4a^2$.
(2) (i) $\frac{1}{2}$, $4\frac{1}{2}$; (ii) 10, -29; (iii) 7, 1; 1, 7.
(3) 66, 22. (4) $9d$.
(5) $-61\frac{1}{2}$, -565; $6\frac{1255}{2592}$, $27\frac{1009}{2592}$; $-\frac{4}{729}$, $8\frac{728}{729}$, 9.
(6) 1, -1; or 11, 19. (7) 34.
(8) 230; $246\frac{87}{128}$; $1\frac{599}{3125}$, $1\frac{1}{4}$.
(9) 6, 8, 12. (10) 200 miles.
(11) (i) 0, 0; or $-3\frac{1}{2}$, $1\frac{2}{5}$; (ii) $\frac{11\pm3\sqrt{5}}{2}$, $1\pm2\sqrt{5}$;
(iii) 10, 10; or 26, 1.
(12) 21 feet. (13) $x^2-2xy-4y^2$.
(14) $104\frac{5}{6}$; $22\frac{45043}{62500}$, $28\frac{3}{4}$; -15.
(15) B £48 1*s*. 6*d*. more than A.
(17) 1235520, 1716. (18) 32, 44.
(19) $y^6-30y^5+375y^4-2500y^3+9375y^2-18750y+15625$.
(20) 1*tet*3. (21) $\frac{7}{10}z$. (22) $4+5x+x^2$.
(23) 3, $-\frac{4}{19}$. (24) 18, $68\frac{11}{32}$, -3.
(25) 7315. (26 (43·113. (27) $\frac{g^2}{a}$.
(28) (i) 6, 1; (ii) $x=1\frac{19}{28}$, $1\frac{8}{39}$, $\frac{47}{50}$, . . . $y=3\frac{5}{14}$, $3\frac{8}{13}$, $3\frac{19}{25}$, . . .
(29) 42504, 120. (30) $\frac{1-23x^{12}}{1-x}+\frac{2x(1-x^{11})}{(1-x)^2}$.
(31) 150 miles. (32) $8\frac{1}{4}$.
(33) $\pm\frac{1}{2}$, $\pm\frac{2}{3}$. (35) 15, 7.
(36) -1, ±3, 5. (37) -1, $\pm\frac{1}{2}$, $\frac{1}{5}$.

(38) $\frac{2Ss^2}{S^2+s^2}$, $\frac{S^2-s^2}{S^2+s^2}$. (39) 1 hr. $43\frac{1}{8}'$.

(40) 40, 100, 250. (41) 3, -4, 5; 3, -1, 7, -4.

(42) 8000. (43) 215.

(44) 3, -1; or -1, 3. (45) 5; 5. (46) 255.

(47) $12x^3y+\frac{y^3}{4}$; $192x^5y+40x^3y^3+\frac{3xy^5}{4}$.

(50) $20n^2q : m^2p$.

Ex. 53.

(1) $\sqrt{75}$, $\sqrt{a^4b^3c}$, $\sqrt[3]{x^5y^2}$, $\sqrt[4]{81x^8y^9}$.

(2) $xy^2\sqrt{z}$, $2a\sqrt{2ab}$, $3ay\sqrt[3]{2ax^2}$, $4ac\sqrt[4]{5ab^2c^2}$.

(3) $6\sqrt{10}$, $14\sqrt[3]{12}$, $10\sqrt{3}$, $20\sqrt[4]{90}$.

(4) $ab\sqrt{xy}$, $x\sqrt{ab}$, $b^2x\sqrt[3]{2a^2}$, $5a^2by\sqrt[4]{3b}$.

(5) $\frac{1}{2}\sqrt{2a}$, $\frac{1}{z}\sqrt[3]{2xy^2z^2}$, $\frac{1}{5}\sqrt[3]{20}$, $\frac{1}{2b}\sqrt[4]{8ab}$, $\frac{a}{2cy^2}\sqrt{3bcxy}$.

Ex. 54.

(1) $3\sqrt{2}$, $2\sqrt[3]{3}$, $\frac{5}{2}\sqrt[4]{4}$. (2) $2ax\sqrt[6]{72ab^5}$, $a\sqrt[20]{a^3x^9y^{19}}$.

(3) $\frac{3}{a^2}\sqrt[6]{2a^5b}$, $\frac{1}{b}\sqrt[12]{8a^{11}}$.

Ex. 55.

(1) $(a-2b+c)\sqrt{ab}$; $4\sqrt{3}$.

(2) $23-16\sqrt{2}$; $6ab-6bc+5b\sqrt{ac}$.

(3) $\sqrt{7}-\sqrt{5}$; $2\sqrt{5}+\sqrt{6}$; $a+\sqrt{b}$; $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$.

(4) $5\sqrt{2}-6$; $30+12\sqrt{6}$; $\frac{a}{b-c}(\sqrt{b}+\sqrt{c})$;

$\frac{2(x-y)\sqrt{xy}+3xy}{xy-4y^2}$.

(5) $2+\sqrt{3}$. (6) $2\sqrt{3}-2\sqrt{2}$. (7) $2\sqrt{3}+\sqrt{5}$.

(8) $\sqrt{6}-\sqrt{3}$. (9) $3-\sqrt{2}$.

(10) $\sqrt{a+b}+\sqrt{a-b}$. (11) $\sqrt{2a-b}-\sqrt{a-2b}$.

(12) $b-\sqrt{a^2-b^2}$. (13) 5.

(14) $\frac{2\sqrt{2}-\sqrt{6}}{2}$. (15) $\frac{21+12\sqrt{2}}{7}$.

Ex. 56.

(1) 86. (2) $1\frac{1}{12}$. (3) 0, 4. (4) 6.
(5) $\frac{1}{2}, -\frac{3}{2}$. (6) 0. (7) 12, 3. (8) $\frac{2}{3}$, $2\frac{8}{21}$.

Ex. 57.

(1) $\frac{1}{a^2}$, $\frac{3}{xy^3}$, $\frac{6y}{x^5}$, $\frac{4a^2x^4}{b^3y^5}$, $\frac{2b^2xy^3}{3a}$.

(2) $\sqrt[3]{a^2}$, $\sqrt[4]{x^3}$, $\sqrt[6]{a^3b}$, $4\sqrt[6]{xy^5}$, $7m\sqrt[5]{n^3}$, $\frac{1}{\sqrt{a}}$, $\frac{3}{\sqrt[3]{b^2}}$, $4\sqrt[3]{\frac{x}{y^2}}$.

(3) $x^{\frac{3}{2}}$, $xy^{\frac{2}{3}}$, $3a^{\frac{5}{4}}$, $a^{\frac{1}{2}}b^{\frac{1}{3}}$, $5ab^{\frac{1}{2}}c^{\frac{3}{2}}x^{2}$.

(5) x^2y^2; $\frac{y^2}{x^2}$. (6) $x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

(7) $4x$, $9a^{\frac{4}{3}}b^{\frac{1}{2}}$, $16a^2b^{-2}$, $a-2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$, a^2+2+a^{-2},
$4ab^{\frac{2}{3}}-4b+a^{-1}b^{\frac{4}{3}}$.

(8) $a^{\frac{2}{3}}+a^{\frac{1}{2}}b^{\frac{1}{6}}-a^{\frac{1}{6}}b^{\frac{1}{2}}-b^{\frac{2}{3}}$. (9) $-\frac{4}{a(a-b)}$.

(10) 4, 6, 10, 4, $\frac{1}{4}$, $\frac{3}{32}$.

(11) $a^{\frac{3}{2}}-3ab^{\frac{1}{3}}+3a^{\frac{1}{2}}b^{\frac{2}{3}}-b$; $16x^{-4}+32x^{-2}+24+8x^2+x^4$;
$a^6b^{-6}-6a^5b^{-4}y^{-1}+15a^4b^{-2}y^{-2}-20a^3y^{-3}+15a^2b^2y^{-4}$
$-6ab^4y^{-5}+b^6y^{-6}$

(12) $2a^{\frac{1}{2}}b^{\frac{1}{2}}+11b$.

(13) $2a^{\frac{11}{6}}+a^{\frac{5}{3}}b^{\frac{1}{6}}+a^{\frac{3}{2}}b^{\frac{1}{3}}-11a^{\frac{4}{3}}b^{\frac{1}{2}}-5a^{\frac{7}{6}}b^{\frac{2}{3}}-12ab^{\frac{5}{6}}$.

(14) $ab^{-1}+3+8a^{-1}b-8a^{-3}b^3$.

(15) $3x^{-2}-3x^{-1}y^{\frac{1}{2}}+y$. (16) $2x+1-3x^{-1}$.

(17) $6a^{\frac{1}{3}}x^{\frac{1}{2}}$; $\frac{2}{3}a^{\frac{1}{2}}b^{\frac{2}{3}}x^{\frac{1}{12}}$.

(18) $2^{\frac{2}{3}}\cdot 3^{\frac{1}{3}}$, $2^{\frac{3}{4}}\cdot 3^{\frac{1}{2}}$, $2^{\frac{5}{6}}\cdot 3^{\frac{1}{6}}$, $2^{\frac{3}{4}}$; 24.

(19) $63(\sqrt{2}+1)$; $\frac{a(b^8-256)}{b^7(b+2)}$. (20) $2c\sqrt[3]{a^2b}$.

(21) $x=4$. (22) $x=8$.

(23) $x=2\sqrt[3]{18}$. (24) $x=2\cdot 5$ or $\cdot 4$

(25) $x=\pm 2$ or $\pm\frac{1}{2}$. (26) $x=3\frac{3}{8}$.

(27) $x=9$, $y=8$. (28) $x=a^2$ or $4a^2$.

(29) $x=0$ or 9. (30) $x=\frac{1}{\sqrt[n]{2}}$ or $\frac{1}{\sqrt[n]{-3}}$.

Ex. 58.

(1) $\sqrt{p^2-4q}$. (2) $-p\sqrt{p^2-4q}$.
(3) $-pq$. (4) $-p(p^2-3q)$.
(5) $p^4-4p^2q+2q^2$. (6) $q(p^2-2q)$.
(7) 28, -144; $\frac{7}{9}$, $\frac{19}{27}$.
(8) *Rational, unequal, opposite in sign.*
(9) *Surd, same sign.*
(10) *Impossible.*
(11) *Rational, unequal, opposite in sign.*
(12) *Equal, opposite in sign.*
(13) *Equal, same sign* $(-)$.
(14) $x^2-3x+2=0$. (15) $x^2-4x-21=0$.
(16) $6x^2-5x+1=0$. (17) $2x^2+11x+5=0$.
(18) $x^2-\cdot 8x+\cdot 15$. (19) $63x^2-32x-63=0$.
(20) $x^2-4ax+3a^2-4ab-4b^2=0$.

Ex. 59.

(1) $(x+13)(x-17)$. (2) $(3x-2)(2x-15)$.
(3) $(2x+7)(7x-1)$. (4) $(x-2)(10x+9)$.
(5) $(x-2a+b)(x+3a-b)$. (6) $(x-y-1)(x+3y-4)$.
(7) $(x-1+a)(x-a-a^2)$.
(8) $(a^2x-b^2y-c^2)(bx+ay)$.

Ex. 60.

(2) -486. (3) 1555.

Ex. 61.

(1) $\frac{5}{4}$, *max.* (2) $\frac{7}{4}$, *min.* (3) $\frac{1}{4}(a-b)^2$, *max.*
(4) $\frac{1}{2}$, *max.*, $-\frac{1}{2}$, *min.*, *i.e.* all values lie between $\frac{1}{2}$ and $-\frac{1}{2}$.
(5) 2, *min.*, -2, *max.*; *i.e.* no values between 2 and -2.
(6) $\frac{4ab}{(a+b)^2}$, *min.* unless $x=0$.
(7) $\frac{a}{2}$, $\frac{a}{2}$. (8) $\frac{1}{4}$.
(9) Each side $=\frac{a}{\sqrt{2}}$. (10) Each side $= s-{}^{a}$.

Ex. 62.

(1) $(ar-cp)^2 = (aq-bp)(br-cq)$.

(2) $a^2 = b+2c$. (3) $a^4+4a^2c = b^2$.

(4) $(k+h)a^4 = (k-h)b^4$. (5) $a^3-3ab^2+2c^3 = 0$.

(6) $a^4b^4+a^4c^4+b^4c^4 = a^2b^2c^2$.

(7) $a^3+b^3+c^3 = 3abc$, i.e.

$a+b+c = 0$, or $a^2+b^2+c^2 = ab+ac+bc$.

(8) $a^2+b^2+c^2 = \pm 2ab\sqrt{2}$. (9) $l^2+m^2+n^2 = 2lmn+1$.

(10) $x^3+y^3+z^3+xyz = 0$.

Ex. 63.

(1) $-1, \frac{5}{2}, \frac{3\pm\sqrt{-15}}{4}$. (2) $\frac{b}{a}+\frac{1}{2}, \frac{b}{a}-\frac{9}{2}$.

(3) $1, -\frac{1}{2}$. (4) ± 4. (5) $5, -2\frac{1}{2}$.

(6) $2, -3, \frac{1}{2}(-1\pm\sqrt{5})$. (7) $4, -3$.

(8) $4, -6$. (9) $-1, \frac{1}{2}(1\pm\sqrt{-3})$.

(10) $\frac{2}{3}, -\frac{3}{2}, \frac{1}{12}(-13\pm\sqrt{313})$.

(11) $6\frac{29}{100}$. (12) $\frac{1}{4}, \frac{1-\sqrt{33}}{8}$.

(13) $1\frac{41}{48}$. (14) $\frac{1}{3}, 0$.

(15) $\pm 2\sqrt{2}, \pm\frac{1}{2\sqrt{-2}}, \pm\frac{1}{8}(5\pm\sqrt{41})^{\frac{3}{2}}$.

(16) $x = \frac{1}{4}(2a+2b+2-\sqrt{4a+1}-\sqrt{4b+1})$;
$y = \frac{1}{4}(2a-2b-\sqrt{4a+1}+\sqrt{4b+1})$.

(17) $x = 5, -8\frac{1}{5}, \frac{1}{5}(4\pm 3\sqrt{109})$,
$y = -2, 6\frac{4}{5}, \frac{2}{5}(-3\mp\sqrt{109})$.

(18) $x = 2, 6, \frac{1}{2}(-9\pm\sqrt{33})$; $y = 6, 2, \frac{1}{2}(-9\mp\sqrt{33})$.

(19) $x = 3, 2, \frac{\pm\sqrt{61}+5}{2}, \pm\sqrt{10}-1, \pm\sqrt{-5}-1$,

$y = -2, -3, \frac{\pm\sqrt{61}-5}{2}, \pm\sqrt{10}+1, \pm\sqrt{-5}+1$.

(20) $x = \frac{c}{a+b}+\frac{(a+3b)c^2}{4(a^2-b^2)}, \frac{c}{a+b}+\frac{(9a-5b)c^2}{4(a^2-b^2)}$,
$y = \frac{c}{a+b}-\frac{(3a+b)c^2}{4(a^2-b^2)}, \frac{c}{a+b}+\frac{(5a-9b)c^2}{4(a^2-b^2)}$.

Ex. 64.

(7) *Former*. (8) *Latter*.
(9) *Former*. (10) *Former*.

Ex. 65.

(1) $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\ldots$

(2) $a^{\frac{3}{4}}\left\{1+\frac{3}{4}\frac{x}{a}-\frac{3}{32}\frac{x^2}{a^2}+\frac{5}{128}\frac{x^3}{a^3}-\ldots\right\}$

(3) $1+4x+10x^2+20x^3+\ldots$

(4) $a^5-\frac{5}{2}a^3x^2+\frac{15}{8}ax^4-\frac{5}{16}a^{-1}x^6+\ldots$

(5) $x^{-3}-\frac{3}{2}x^{-4}y+\frac{15}{8}x^{-5}y^2-\frac{35}{16}x^{-6}y^2+\ldots$

(6) $(2x)^{-\frac{1}{4}}\{1+\frac{3}{8}x^{-1}y+\frac{45}{128}x^{-2}y^2+\frac{405}{1024}x^{-3}y^3+\ldots\}$

(7) $1-x-2x^2-6x^3-\ldots$

(8) $a^{-\frac{1}{2}}\left\{1+\frac{x}{2a}+\frac{5x^2}{8a^2}+\frac{15x^3}{16a^3}+\ldots\right\}$

(9) $1+\frac{5}{2}y+\frac{55}{8}y^2+\frac{935}{48}y^3+\ldots$

(10) $1+\frac{2}{3}x+\frac{5}{9}x^2-\frac{14}{81}x^3+\ldots$

(11) $1-\frac{3}{2}x+\frac{15}{8}x^2-\frac{11}{16}x^3+\ldots$

(12) $(-1)^r\frac{1.3.5\ldots\ldots(2r-5)}{\underline{\vert r-1}.2^{r-1}}a^{\frac{3}{2}-r}x^{r-1}$.

(13) $\frac{r(r+1)}{1.2}a^{-r-2}x^{r-1}$.

(14) 8·06225; 1·01098. (15) 2·002224.

(16) $1+\frac{4}{5}x-\frac{2}{25}x^2-\frac{196}{125}x^3+\frac{119}{625}x^4-\ldots$

(17) 351; $2r-1$. (18) ·100033 . . .

(20) $1+n+\frac{n(n+1)}{1.2}+\ldots+\frac{n(n+1)\ldots(n+r-1)}{\underline{\vert r}}$;

$\frac{(n+1)(n+2)\ldots(n+r)}{\underline{\vert r}}$.

Ex. 66.

(1) $1+2x+x^2-4x^3$; $5-17x+56x^2-185x^3$;

$\frac{3}{4}+\frac{x}{16}+\frac{3x^2}{64}+\frac{9x^3}{256}$.

(2) $y-2y^2+5y^3-14y^4+\ldots$; $y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\ldots$.

(3) $\frac{1+2x}{1-x+x^2}$, $\frac{3-7x}{1-3x+x^2}$, $\frac{3}{4+x-5x^2}$.

(4) $\frac{2}{x-3}+\frac{1}{x-2}$; $\frac{6}{x+3}-\frac{6}{x+4}$; $\frac{-1}{3(2x-1)}+\frac{8}{3(x-5)}$;

$\frac{4}{7(x+2)}+\frac{3}{7(x-5)}$.

(5) $\frac{3}{4x}-\frac{1}{8(x-2)}+\frac{3}{8(x+2)}$; $\frac{4}{25x}-\frac{4}{5x^2}+\frac{71}{25(x+5)}$;
$\frac{11}{3(x-1)}+\frac{2}{(x-1)^2}+\frac{10}{3(x+2)}$; $\frac{5}{x-1}-\frac{5}{x}$;
$\frac{10}{3(x+1)}-\frac{4x+7}{3(x^2-x+1)}$.

Ex. 67.

(1) 1, 2, 3, ·5, 1·5, 3·5, −·5, −2, −2·5, −·125

(2) 256, 16, 4, 1024, $\frac{1}{16}$, $\frac{1}{4}$, $\frac{1}{2}$, $\sqrt[4]{2}$.

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Ex. 68.

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(2) ·434294, ·868588.

(3) 2·1789769.

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(5) 1·9138139, 1·8976271.

Ex. 69.

(1) $\frac{1}{20}$, $\frac{1}{25}$, ·03, ·055; $\frac{21}{20}$, $\frac{26}{25}$, 1·03, 1·055; 2, 6, $5\frac{5}{9}$; $16\frac{2}{3}$, $4\frac{1}{2}$, $8\frac{1}{3}$.

(2) (i) $\frac{B-A}{nA}$, $\frac{100(B-A)}{nA}$;

(ii) $\sqrt[n]{\frac{B}{A}}-1$, $100\left(\sqrt[n]{\frac{B}{A}}-1\right)$.

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(7) 5·358 . . per cent.

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Ex. 71.

(1) $\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2}$, $\frac{12}{29}$;

$\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2}$, $\frac{11}{14}$; $2+\frac{\dot{1}}{4}$, $\frac{682}{305}$;

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Ex. 72.

(2) $\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}$.

(3) $\frac{a}{x}$. (4) x^3-x-1.

(5) (i) $\pm\sqrt{ab}$; (ii) $\pm\sqrt{b^2-a^2}$.

(6) $x=\frac{a(a-m)(a-n)}{l(l-m)(l-n)}$, $y=\frac{a(a-l)(a-n)}{m(m-l)(m-n)}$,

$z=\frac{a(a-l)(a-m)}{n(n-l)(n-m)}$.

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(15) 3, 6, 12.

(16) $x^{\frac{7}{2}}-7x^3y^{\frac{1}{2}}+21x^{\frac{5}{2}}y-35x^2y^{\frac{3}{2}}+35x^{\frac{3}{2}}y^2-21xy^{\frac{5}{2}}$
$+7x^{\frac{1}{2}}y^3-y^{\frac{7}{2}}$;

$x^{\frac{2}{7}}-\frac{1}{7}x^{-\frac{12}{7}}y^2-\frac{8}{49}x^{-\frac{26}{7}}y^4-\frac{18}{343}x^{-\frac{40}{7}}y^6$;

$-\frac{6.13\ .\ .\ .\ .\ (7r-8)}{7^r\ .\ \lfloor\underline{r}}x^{\frac{2-14r}{7}}y^{2r}$.

(17) $1-2x+4x^2-8x^3$; $\sqrt[3]{a^2}-\sqrt[3]{ax}+\sqrt[3]{x^2}$.

(18) $\dfrac{(1-2a)(1+2a+4a^2)}{a^3}$, $\dfrac{2(1-6x)(1-2x)}{x^2}$,

$\dfrac{2(4m-n)}{(m+n)^2(m-n)}$.

(19) $1+2x^{-\frac{1}{3}}-3x^{-\frac{2}{3}}+4x^{-1}$; $3+2\sqrt{2}$.

(20) $-\dfrac{b+c}{(1+b^2)(1+c^2)}$.

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